MATH 565 Monte Carlo Methods in Finance

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In-Class Final **Tuesday, November 30**

Instructions:

i. *This in-class part of the final exam consists of TWO questions. Answer both of them.*

ii. *The time allowed for this test is 120 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *Calculators, even of the programmable variety, are allowed. Computers, but only using MATLAB or JMP, are also allowed. No internet access.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (25 marks)
Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be independent and identically distributed uniform random variables on $[0,1]^2$, where $\boldsymbol{X}_i = (X_{i1}, X_{i2})$.

a) Consider

$$Y_n = \frac{1}{n}\sum_{i=1}^{n}(X_{i1} + X_{i2}^2).$$

Compute $E(Y_n)$ and $\mathrm{var}(Y_n)$ analytically.

Answer:

$$\begin{aligned}
E(Y_n) &= \frac{1}{n}\sum_{i=1}^{n} E(X_{i1} + X_{i2}^2) = \frac{1}{n}\sum_{i=1}^{n}\left[E(X_{i1}) + E(X_{i2}^2)\right] = E(X_{11}) + E(X_{12}^2) \\
&= \int_0^1 x\,\mathrm{d}x + \int_0^1 x^2\,\mathrm{d}x = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}, \\
\mathrm{var}(Y_n) &= \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}(X_{i1} + X_{i2}^2) = \frac{1}{n^2}\sum_{i=1}^{n}[\mathrm{var}(X_{i1}) + \mathrm{var}(X_{i2}^2)] = \frac{1}{n}[\mathrm{var}(X_{11}) + \mathrm{var}(X_{12}^2)] \\
&= \frac{1}{n}\left[\int_0^1 (x-1/2)^2\,\mathrm{d}x + \int_0^1 (x^2 - 1/3)^2\,\mathrm{d}x\right] \\
&= \frac{1}{n}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4} + \frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\right]\Bigg|_0^1 = \frac{31}{180n}
\end{aligned}$$

b) Let $\boldsymbol{Z}_i = (1 - X_{i1}, 1 - X_{i2})$ for $i = 1, 2, \ldots$, and let

$$W_n = \frac{1}{2n}\sum_{i=1}^{n}\left[(X_{i1} + Z_{i1}) + (X_{i2}^2 + Z_{i2}^2)\right]$$

Compute $E(W_n)$ and $\mathrm{var}(W_n)$ analytically.

Answer: Note that from the definition

$$\begin{aligned}
W_n &= \frac{1}{2n}\sum_{i=1}^{n}\left[1 + X_{i2}^2 + (1 - X_{i2})^2\right] = \frac{1}{2n}\sum_{i=1}^{n}\left[2 - 2X_{i2} + 2X_{i2}^2\right] \\
&= 1 + \frac{1}{n}\sum_{i=1}^{n}\left[-X_{i2} + X_{i2}^2\right] \\
E(W_n) &= 1 + \frac{1}{n}\sum_{i=1}^{n} E(-X_{i2} + X_{i2}^2) = 1 + \frac{1}{n}\sum_{i=1}^{n}[-E(X_{i2}) + E(X_{i2}^2)] = 1 - E(X_{12}) + E(X_{12}^2) \\
&= 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}, \\
\mathrm{var}(W_n) &= \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}(-X_{i2} + X_{i2}^2) = \frac{1}{n}\,\mathrm{var}(X_{12}^2 - X_{12}) \\
&= \frac{1}{n}\int_0^1 (x^2 - x + 1/6)^2\,\mathrm{d}x = \frac{1}{n}\int_0^1\left(x^4 - 2x^3 + \frac{4x^2}{3} - \frac{x}{3} + \frac{1}{36}\right)\mathrm{d}x \\
&= \frac{1}{n}\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{4x^2}{9} - \frac{x^2}{6} + \frac{x}{36}\right]\Bigg|_0^1 = \frac{1}{180n}
\end{aligned}$$

c) Note that Y_n is a simple Monte Carlo estimator for

$$\int_{[0,1]^2} (x_1 + x_2^2)\,\mathrm{d}\boldsymbol{x},$$

and W_n is an antithetic variate estimator for the same. Compute the mean square errors of Y_{2n} and W_n. Which is smaller?

Answer: Both Y_n and W_n are unbiased. Thus, their mean square errors are their variances. Since $\mathrm{var}(Y_{2n}) = 31/(360n)$ and $\mathrm{var}(W_n) = 1/(180n)$, W_n has the smaller mean squared error.

2. (25 marks)
Consider two stocks whose prices, S_1 and S_2, are modeled as follows:

$$\begin{aligned}
S_1(t) &= S_1(0)\exp\left(\left(r - \frac{\sigma_1^2}{2}\right)t + \sigma_1\sqrt{t}(0.8X_1 + 0.6X_2)\right), \\
S_2(t) &= S_2(0)\exp\left(\left(r - \frac{\sigma_2^2}{2}\right)t + \sigma_2\sqrt{t}(0.8X_1 - 0.6X_2)\right), \\
&X_1, X_2 \text{ i.i.d. } N(0,1).
\end{aligned}$$

The basket call option has a payoff of

$$\max\left(\max(S_1(T), S_2(T)) - K, 0\right) e^{-rT},$$

where T is the time to expiry. Price this option using Monte Carlo with a relative error of 1% or less, assuming $T = 1$, $r = 1\%$, $S_1(0) = S_2(0) = K = 100$, $\sigma_1 = 50\%$, and $\sigma_2 = 30\%$. *For up to 10 points extra credit, compute the probability that $S_1(T) > S_2(T)$.*

Answer: The MATLAB program that solves this problem is

```
%% Problem 2 on In Class Final
r=0.01; %interest rate
S01=100; %initial asset price
S02=100; %initial asset price
sig1=0.5; %volatility of stock 1
sig2=0.3; %volatility of stock 2
K=100; %strike price
T=1; %time to expiry
n=1e5; %number of samples
Xmat=randn(n,2); %get normal random numbers
ST=[S01*exp((r-sig1^2/2) + sig1*(Xmat*[0.8;0.6])) ...
    S02*exp((r-sig2^2/2) + sig2*(Xmat*[0.8;-0.6]))]; %stock paths
payoff=max(max(ST,[],2)-K,0)*exp(-r*T); %payoff
call=mean(payoff); %approximate call price
err=1.96*std(payoff)/sqrt(n); %estimate of error
rerr=err/call; %estimate of relative error
disp(' ')
disp(['Using ' int2str(n) ' samples'])
disp('The price of the basket call option')
disp(['     is $' num2str(call) ' +/- ' num2str(err)])
disp(['     for a relative error of +/- ' num2str(100*rerr) '%'])

%%Extra credit
p=mean(ST(:,1)>ST(:,2)); %estimate of proportion that 1 is bigger
errp=1.96*sqrt(p*(1-p))/sqrt(n); %estimate of error
disp('The probability that stock 1 has a higher price is')
disp(['     is ' num2str(p) ' +/- ' num2str(errp)])

Using 100000 samples
The price of the basket call option
     is $27.4579 +/- 0.25105
     for a relative error of +/- 0.9143%
The probability that stock 1 has a higher price is
     is 0.43702 +/- 0.0030743
```