MATH 565 Monte Carlo Methods in Finance

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In-Class Final **Thursday, December 1**

Instructions:

i. *This in-class part of the final exam consists of TWO questions for 50 points possible. Answer both questions.*

ii. *The time allowed for this test is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *Calculators, even of the programmable variety, are allowed.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (30 points)
Consider the problem of approximating

$$\mu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(x^2 + y^2)\, e^{-(x^2+y^2)/2}\, \mathrm{d}x\, \mathrm{d}y.$$

a) Describe explicitly the simple Monte Carlo method (i.i.d. sampling) that you would use to approximate this integral. What probability distribution would you choose for the random numbers, and how would you use the random numbers to approximate the integral?

Answer: The integral may be written as

$$\mu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2\pi \cos(x^2 + y^2)\, \frac{1}{2\pi} e^{-(x^2+y^2)/2}\, \mathrm{d}x\, \mathrm{d}y,$$

where now the second term is the bivariate normal probability distribution function. Thus, the Monte Carlo approximation is

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} 2\pi \cos(X_i^2 + Y_i^2),$$

where the $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ *are i.i.d. standard Gaussian random variables.*

b) How would you determine the sample size, n, required to obtain an approximation that has an absolute error of no more than 0.005 with probability $\approx 95\%$ or more?

Answer: Take a sample of standard Gaussian random vectors $(X_1, Y_1), \ldots, (X_{n_0}, Y_{n_0})$*, for say,* $n_0 = 1000$*. Use that sample to compute the sample mean as above,* $\hat{\mu}_{n_0}$*, and the sample variance,*

$$\hat{\sigma}_{n_0}^2 = \frac{1}{n_0 - 1} \sum_{i=1}^{n_0} [2\pi \cos(X_i^2 + Y_i^2) - \hat{\mu}_{n_0}]^2$$

Then applying the Central Limit Theorem, estimate the sample, n*, needed as*

$$n = \left\lceil 1.5 \left(\frac{1.96 \hat{\sigma}_{n_0}}{0.005} \right)^2 \right\rceil,$$

where 1.5 is a fudge factor. Using a new random sample of size n*, the sample mean,* $\hat{\mu}_n$*, should now give the desired accuracy.*

c) Now suppose that you wish to use Latin hypercube sampling or Sobol' sampling (your choice). Describe how the algorithm would change. *(You do not need to describe error estimation. You do not need to describe how you would construct the Sobol' points, but you do need to describe how you would construct the Latin hypercube points.)*

Answer: Given the sample size n, for Latin hypercube points let

$$W_{i1} = \frac{i - U_{i1}}{n}, \qquad W_{i2} = \frac{V(i) - U_{i2}}{n},$$

where the U_{ij} are independent uniform random numbers on $[0,1]$, and $V(\cdot)$ is a random permutation of $1, \ldots, n$. For Sobol' points generate a sample of dimension 2 with n vectors, e.g., $\mathtt{W} = \mathtt{net}(\mathtt{sobolset}(2), \mathtt{n})$. *In either case, these vectors emulate the uniform distribution. To get standard normal looking vectors, we must take the inverse normal cumulative distribution function:*

$$X_i = \Phi^{-1}(W_{i1}), \qquad Y_i = \Phi^{-1}(W_{i2}),$$

where Φ is the standard Gaussian cumulative distribution function.

2. (20 points)
Consider the problem of pricing an Asian Arithmetic Mean call option with a strike price of \$120, which expires in $T = 1$ year. Assume that the interest rate is 0. This means that the (discounted) payoff is

$$\max\left(\int_0^1 S(t)\,\mathrm{d}t - 120, 0\right).$$

Assume that the stock price today is $S(0) = 100$, and that each one of n stock paths is generated at d time steps. Here is *one* stock path generated for $d = 16$:

t	0	0.0625	0.125	0.1875	0.25	0.3125	0.375	0.4375	0.5
$S(t)$	100	106	79	82	77	120	101	109	119

t	0.5625	0.625	0.6875	0.75	0.8125	0.875	0.9375	1
$S(t)$	116	137	148	161	170	185	190	211

a) Estimate the Asian Arithmetic Mean call option payoff for the path above using all $d = 16$ time steps.

Answer: A right rectangle rule gives

$$\mathit{payoff} = \max\left(\frac{1}{d}\sum_{j=1}^{d} S(j/16) - 120, 0\right) = \max(132 - 120, 0) = 12.$$

Alternatively, a trapezoidal rule gives

$$\begin{aligned}\mathit{payoff} &= \max\left(\frac{1}{d}\left[\frac{S(0)}{2} + S(1/16) + \cdots + S(15/16) + \frac{S(1)}{2}\right] - 120, 0\right)\\ &= \max(129 - 120, 0) = 9.\end{aligned}$$

Either approach is okay.

b) Estimate the Asian Arithmetic Mean call option payoff for the path above now only using $d = 8$ time steps (but still $T = 1$).

Answer: A right rectangle rule gives

$$\textit{payoff} = \max\left(\frac{1}{d}\sum_{j=1}^{d} S(j/8) - 120, 0\right) = \max(134 - 120, 0) = 14.$$

Alternatively, a trapezoidal rule gives

$$\begin{aligned}\textit{payoff} &= \max\left(\frac{1}{d}\left[\frac{S(0)}{2} + S(1/8) + \cdots + S(7/8) + \frac{S(1)}{2}\right] - 120, 0\right)\\ &= \max(127 - 120, 0) = 7.\end{aligned}$$

c) What is the difference in the payoffs using the two different values of d?

Answer: There is a difference of \$2.

d) If the option is priced using only $d = 8$ instead of $d = 16$ time steps, but for the same number of paths, n, how does the computational time required change?

Answer: The time will be cut in half.

Note that this highlights a trade-off. Increasing d *will give a more accurate approximation to the integral, but take more time. Increasing* n *will decrease the sampling error but also increase the time. The total time taken is approximately proportional to* nd.