MATH 565 Monte Carlo Methods in Finance

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Take-Home Final **Due 12 noon, Tuesday, December 7**

Instructions:

i. *This take-home part of the final exam consists of TWO questions for a total possible of* 50 marks. *Answer both of them.*

ii. *You may consult any book, web page, software repository or other inanimate object. You may use the m-files on Blackboard. You* may not *consult any other person face-to-face, by telephone, by email, or by any other means.*

iii. *Show all your work to justify your answers. Submit hard copies of your derivations, programs, output, and explanations. Answers without adequate justification will not receive credit.*

1. (25 marks)
Let $\boldsymbol{X} = (X_1, X_2)$ be a uniform random vector on $[0,1]^2$. One wishes to use $\boldsymbol{X}$ to construct a Gaussian random vector, $\boldsymbol{Z}$, with zero mean and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Explain how to do this.

Answer: First use the inverse cumulative distribution function to obtain standard Gaussian random variables, $\boldsymbol{T}$:

$$T_1 = \Phi^{-1}(X_1), \quad T_2 = \Phi^{-1}(X_2), \qquad \textit{where } \Phi(t) = \int_{-\infty} \frac{1}{\sqrt{2\pi}} e^{-v^2/2}\,\mathrm{d}v.$$

Note that if $\boldsymbol{Z} = \mathsf{A}\boldsymbol{T}$, then $E[\boldsymbol{Z}] = \mathsf{A}E[\boldsymbol{T}] = \boldsymbol{0}$ and

$$\operatorname{cov}(\boldsymbol{Z}) = E[\boldsymbol{Z}\boldsymbol{Z}^T] = E[\mathsf{A}\boldsymbol{T}\boldsymbol{T}^T\mathsf{A}^T] = \mathsf{A}\mathsf{A}^T.$$

We need to find A such that $\mathsf{A}\mathsf{A}^T = \Sigma$. This can be done using the Cholesky decomposition, in which case

$$\mathsf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

2. (25 marks)
Consider two stocks whose prices, S_1 and S_2, are modeled as follows:

$$S_1(t+\Delta) = S_1(t)\exp\left(\left(r - \frac{\sigma_1^2}{2}\right)\Delta + \sigma_1\sqrt{\Delta}(0.8X_1 + 0.6X_2)\right),$$
$$S_2(t+\Delta) = S_2(t)\exp\left(\left(r - \frac{\sigma_2^2}{2}\right)\Delta + \sigma_2\sqrt{\Delta}(0.8X_1 - 0.6X_2)\right),$$
$$X_1, X_2 \text{ i.i.d. } N(0,1).$$

The basket Asian call option has a payoff of

$$\max\left(\frac{1}{d}\sum_{j=1}^{d}\frac{1}{2}\left[S_1(jT/d) + S_2(jT/d)\right] - K, 0\right) e^{-rT},$$

where T is the time to expiry. Price this option using Monte Carlo with a relative error of 1% or less, assuming $T = 1$, $d = 52$, $r = 1\%$, $S_1(0) = S_2(0) = K = 100$, $\sigma_1 = 50\%$, and $\sigma_2 = 30\%$. You must use at least one variance reduction method (control variates, importance sampling, antithetic variates, Latin hypercube sampling, low discrepancy sampling, etc., your choice). Also, compute the probability that

$$\frac{1}{d}\sum_{j=1}^{d} S_1(jT/d) > \frac{1}{d}\sum_{j=1}^{d} S_2(jT/d).$$

Answer: The MATLAB program that solves this problem is

```
%% Problem 2 on Take Home Final by Sobol Sequences
tic, clear all
r=0.01; %interest rate
S01=100; %initial asset price
S02=100; %initial asset price
sig1=0.5; %volatility of stock 1
sig2=0.3; %volatility of stock 2
K=100; %strike price
d=52; %number of time steps
T=1; %time to expiry
n=2^11; %number of samples, power of 2 is better
Delta=T/d;
S1=S01*ones(n,d+1);
S2=S1;
nrep=30;
call=zeros(nrep,1);
p=call;
for k=1:nrep
    pnet=scramble(sobolset(2*d),'MatousekAffineOwen'); %scrambled Sobol
    Xmat=norminv(net(pnet,n)); %sample points
    for j=1:d
        S1(:,j+1)=S1(:,j).*exp((r-sig1^2/2)*Delta + sig1*sqrt(Delta)*(Xmat(:,2*j+[-1 0])*[
        S2(:,j+1)=S2(:,j).*exp((r-sig2^2/2)*Delta + sig2*sqrt(Delta)*(Xmat(:,2*j+[-1 0])*[
    end
    avg1=mean(S1(:,2:d+1),2);
    avg2=mean(S2(:,2:d+1),2);
    payoff=max((avg1+avg2)/2-K,0)*exp(-r*T); %payoff
    call(k)=mean(payoff); %approximate call price
    p(k)=mean(avg1>avg2); %estimate of proportion that 1 is bigger
end
callprice=mean(call);
err=1.96*std(call)/sqrt(nrep); %estimate of error
rerr=err/callprice; %relative error
disp(' ')
disp(['Using ' int2str(nrep) ' replications of ' int2str(n) ' samples each'])
disp('The price of the basket Asian call option')
```

```
disp(['     is $' num2str(callprice) ' +/- ' num2str(err)])
disp(['     for a relative error of +/- ' num2str(100*rerr) '%'])

pavg=mean(p); %estimate of proportion that 1 is bigger
errp=1.96*std(p)/sqrt(nrep); %estimate of error
disp('The probability that path average of stock has a higher price is')
disp(['     is ' num2str(pavg) ' +/- ' num2str(errp)])
toc
```

```
Using 30 replications of 2048 samples each
The price of the basket Asian call option
     is $7.8171 +/- 0.034107
     for a relative error of +/- 0.43631%
The probability that path average of stock has a higher price is
     is 0.45566 +/- 0.0023986
Elapsed time is 3.047010 seconds.
```