MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell **Fall 2007**

In-Class Part of Final Exam **Thursday, December 13**

Instructions:

i. *This test consists of FIVE questions for a total of 50 points possible. Answer all of them.*

ii. *The time allowed for this test is 120 minutes*

iii. *This test is closed book, but you may use 2 double-sided letter-size sheets of notes.*

iv. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (10 marks)
Consider three functions defined as follows:

$$g_1(x) = 100, \qquad g_2(x) = e^x, \qquad g_3(x) = e^{100x}.$$

Let X be a standard Gaussian (normal) random variable, i.e., $X \sim N(0,1)$. Suppose that one estimates the means $\mu_j = E[g_j(X)]$, $j = 1, 2, 3$ by a simple Monte Carlo method:

$$\hat{\mu}_j = \frac{1}{n} \sum_{i=1}^{n} g_j(X_i),$$

where $X_1, \ldots, X_n$ are i.i.d. $N(0,1)$.

a) For which one or more j is $\hat{\mu}_j$ an unbiased estimate of μ_j?

Answer: $E[\hat{\mu}_j] = \mu_j$ *for all* j.

b) For which one or more j is $\hat{\mu}_j = \mu_j$ with probability one?

Answer: Simple Monte Carlo always gives the exact answer for constant functions, so $\hat{\mu}_1 = \mu_1$ *with probability one.*

c) For which j is $\mathrm{MSE}(\hat{\mu}_j) = \mathrm{E}[\hat{\mu}_j - \mu_j]^2$ the largest?

Answer: Since $E[\hat{\mu}_j] = E[g_j(X)]$, *it follows that* $\mathrm{MSE}(\hat{\mu}_j) = E[\hat{\mu}_j - \mu_j]^2 = \mathrm{var}(g_j(X))/n$, *which is largest for* $j = 3$.

2. (10 marks)
The rank-1 lattice point set has the form $P_{\text{lat}} = \{\boldsymbol{z}_i = (i-1)\boldsymbol{c}/n + \boldsymbol{\Delta} \mod 1 : i = 1, \ldots, n\}$, where n is the number of points, $\boldsymbol{c}$ is the d-dimensional generating vector, and $\boldsymbol{\Delta}$ is the d-dimensional shift vector. Let $d = 2$, $n = 8$, and $\boldsymbol{\Delta} = (1/16, 1/16)$. Calculate the two sets P_{lat} corresponding to two possible generating vectors: $(1,3)$ and $(1,4)$. Inspect the sets to determine which generating vector yields a more evenly distributed set on $[0,1]^2$.

Answer: The two sets are given by:

i	$\boldsymbol{z}_i$ *for* $\boldsymbol{c} = (1,3)$	$\boldsymbol{z}_i$ *for* $\boldsymbol{c} = (1,4)$
1	$(1/16, 1/16)$	$(1/16, 1/16)$
2	$(3/16, 7/16)$	$(3/16, 9/16)$
3	$(5/16, 13/16)$	$(5/16, 1/16)$
4	$(7/16, 3/16)$	$(7/16, 9/16)$
5	$(9/16, 9/16)$	$(9/16, 1/16)$
6	$(11/16, 15/16)$	$(11/16, 9/16)$
7	$(13/16, 5/16)$	$(13/16, 1/16)$
8	$(15/16, 11/16)$	$(15/16, 9/16)$

The set using $\boldsymbol{c} = (1,4)$ *only has two different values of* z_2*, and so is worse.*

3. (6 marks)
Rank-1 lattice points, P_{lat}, are used to sample from the uniform distribution on $[0,1]^2$, i.e.,

$$E[g(\boldsymbol{Z})] \approx \frac{1}{8} \sum_{\boldsymbol{z} \in P_{\text{lat}}} g(\boldsymbol{z}), \quad \boldsymbol{Z} = (Z_1, Z_2) \sim U[0,1]^2.$$

Use P_{lat} from the previous problem with the better generating vector to construct a set of points $P_{\text{Gauss}} = \{\boldsymbol{x}_i = (x_{i1}, x_{i2}) : i = 1, \ldots, 8\}$ that may be used to sample from the bivariate standard Gaussian (normal) distribution, i.e.,

$$E[g(\boldsymbol{X})] \approx \frac{1}{8} \sum_{\boldsymbol{z} \in P_{\text{Gauss}}} g(\boldsymbol{z}), \quad \boldsymbol{X} = (X_1, X_2),\ X_1, X_2 \text{ i.i.d. } N(0,1).$$

Note that

x	$-\infty$	-1.534	-1.150	-0.887	-0.674	-0.489	-0.319	-0.157	0
$\Phi(x)$	0	1/16	2/16	3/16	4/16	5/16	6/16	7/16	8/16

x	0.157	0.319	0.489	0.674	0.887	1.150	1.534	∞
$\Phi(x)$	9/16	10/16	11/16	12/16	13/16	14/16	15/16	1

where $\Phi(x) = \int_{-\infty}^{x} e^{-t^2/2}/\sqrt{2\pi}\,\mathrm{d}t$ is the cumulative distribution function of the standard Gaussian distribution.

Answer: The set $P_{\text{lat}} \subset [0,1]^2$*. To obtain vectors that are standard bivariate Gaussian (normal) we take the inverse normal transformation* $\{\boldsymbol{x} = \Phi^{-1}(\boldsymbol{z}) : \boldsymbol{z} \in P_{\text{lat}}\}$*. Using the table above:*

i	$\boldsymbol{z}_i$ *for* $\boldsymbol{c} = (1,3)$	$\boldsymbol{x}_i = \Phi^{-1}(\boldsymbol{z}_i)$
1	$(1/16, 1/16)$	$(-1.534, -1.534)$
2	$(3/16, 7/16)$	$(-0.887, -0.157)$
3	$(5/16, 13/16)$	$(-0.489, 0.887)$
4	$(7/16, 3/16)$	$(-0.157, -0.887)$
5	$(9/16, 9/16)$	$(0.157, 0.157)$
6	$(11/16, 15/16)$	$(0.489, 1.534)$
7	$(13/16, 5/16)$	$(0.887, -0.489)$
8	$(15/16, 11/16)$	$(1.534, 0.489)$

The next three problems consider the discrete time geometric Brownian motion model for a stock price:

$$S(jT/d) = S(jT/d; \boldsymbol{X}) = S(0)\exp((r - \sigma^2/2)jT/d + \sigma\sqrt{T/d}\,(X_1 + \cdots + X_j)), \qquad j = 1, \ldots, d, \quad (1)$$

where the X_j are i.i.d. $N(0,1)$, d is the number of times at which the stock price is monitored, r is the continuously compounded interest rate, $S(0)$ is the initial stock price, T is the ending time, and σ is the volatility of the stock. The discretely monitored Asian arithmetic mean call option with strike price K has a discounted payoff at time T of

$$\max\left(\frac{1}{d}\sum_{j=1}^{d} S(jT/d) - K, 0\right) e^{-rT}. \quad (2)$$

4. (6 marks)

a) Explain briefly in words why the stock price model in (1) has the term $\exp((r - \sigma^2/2)jT/d)$ instead of $\exp(rjT/d)$.

Answer: The term $\exp((r - \sigma^2/2)jT/d)$ is determined by the condition of no arbitrage, i.e., $E[S(jT/d)] = S(0)e^{rjT/d}$.

b) Explain briefly in words why the payoff in (2) has the term e^{-rT}.

Answer: This is a discounting term. The value of money paid at time T is less than the value of the same amount of money now.

5. (10 marks)
Below is a table of points with $\boldsymbol{x}_i, i = 1, \ldots, 8$, and the corresponding $S(1/2; \boldsymbol{x}_i)$ and $S(1; \boldsymbol{x}_i)$ for some values $\boldsymbol{x}_i$. Complete the table by filling in the blanks, and then use this data to estimate the price of the Asian arithmetic mean call option with payoff (2), assuming an asset price model (1) with $d = 2$, $r = 0.05$, $S(0) = 100$, $T = 1$, $\sigma = 0.4$, and $K = 100$.

i	$\boldsymbol{x}_i$	$S(1/2)$	$S(1)$
1	(x_{11}, x_{12})	64	41
2	(x_{21}, x_{22})	77	72
3	$(-0.489, 0.887)$		
4	(x_{41}, x_{42})	94	72
5	(x_{51}, x_{52})	103	106
6	(x_{61}, x_{62})	113	172
7	$(0.887, -0.489)$		
8	(x_{81}, x_{82})	152	172

Answer: Note that these points happen to be the same ones as in P_{Gauss}. The values of $S(1/2; \boldsymbol{x}_i)$ and $S(1; \boldsymbol{x}_i)$ are calculated according to (1) and the discounted payoff according *to* (2)*:*

i	$\boldsymbol{x}_i$	$S(1/2;\boldsymbol{x}_i)$	$S(1;\boldsymbol{x}_i)$	*discounted payoff*
1	$(-1.534,-1.534)$	64	41	0
2	$(-0.887,-0.157)$	77	72	0
3	$(-0.489,0.887)$	86	109	0
4	$(-0.157,-0.887)$	94	72	0
5	$(0.157,0.157)$	103	106	4
6	$(0.489,1.534)$	113	172	40
7	$(0.887,-0.489)$	127	109	17
8	$(1.534,0.489)$	152	172	59

The mean discounted payoff (last column above) is 15*, which is our estimate of the price of the option.*

6. (8 marks)

a) Consider now the following asset price model:

$$S(jT/d) = S(jT/d;\boldsymbol{X}) = S(0)\exp((r-\sigma^2/2)jT/d + \sigma\sqrt{jT/d}\,X_j), \qquad j=1,\dots,d, \quad (3)$$

where again the X_j are i.i.d. $N(0,1)$. Explain why $S(jT/d)$ in (3) has the same distribution as $S(jT/d)$ in (1).

Answer: If $X_1,\dots,X_j$ *are i.i.d.* $N(0,1)$*, then* $X_1+\cdots+X_j \sim N(0,j)$*, which has the same distribution as* $\sqrt{j}X_j$*.*

b) Explain briefly in words why using (3) to perform a Monte Carlo simulation to price the Asian arithmetic mean call option is a *bad* idea.

Answer: Model (3) *does not show the relationship between* $S(0), S(T/d), \dots S(T)$*. It makes them independent random variables, but they are not.*