MATH 565 Monte Carlo Methods in Finance

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In-Class Part of Final Exam **Wednesday, December 3**

Instructions:

i. *This exam consists of FOUR questions for a total of 50 points possible. Answer all of them.*

ii. *The time allowed for this exam is 120 minutes*

iii. *This exam is closed book, but you may use* four *double-sided letter-size sheets of notes.*

iv. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

To solve some of the the problems in this exam you will need the following information. Please refer to it as needed.

Here are sixteen uniformly distributed pseudorandom numbers, x_i, and the respective inverse standard normal distribution of these x_i, i.e., $\Phi^{-1}(x_i)$:

pseudorandom x_i	0.1622	0.7943	0.3112	0.5285	0.1656	0.6020	0.2630	0.6541
$\Phi^{-1}(x_i)$	−0.9855	0.8214	−0.4924	0.0716	−0.9715	0.2585	−0.6342	0.3964

pseudorandom x_i	0.6892	0.7482	0.4505	0.0838	0.2290	0.9133	0.1524	0.8258
$\Phi^{-1}(x_i)$	0.4936	0.6687	−0.1243	−1.3798	−0.7422	1.3616	−1.0263	0.9378

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, \mathrm{d}t$. Here is $\Phi^{-1}(x)$ for some evenly spaced numbers:

x	0	1/16	2/16	3/16	4/16	5/16	6/16	7/16	8/16
$\Phi^{-1}(x)$	$-\infty$	−1.534	−1.150	−0.887	−0.674	−0.489	−0.319	−0.157	0

x	9/16	10/16	11/16	12/16	13/16	14/16	15/16	1
$\Phi^{-1}(x)$	0.157	0.319	0.489	0.674	0.887	1.150	1.534	∞

The discrete time geometric Brownian motion model for a stock price is

$$S(jT/d) = S(jT/d; \boldsymbol{X}) = S(0) \exp((r - \sigma^2/2)jT/d + \sigma \sqrt{T/d}\,(X_1 + \cdots + X_j)), \quad j = 1, \ldots, d, \quad (1)$$

where the X_j are i.i.d. $N(0, 1)$, d is the number of times at which the stock price is monitored, r is the continuously compounded interest rate, $S(0)$ is the initial stock price, T is the ending time, and σ is the volatility of the stock.

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1. (12 points)
Consider the problem of estimating $\mu = E[Y]$, the mean of the random variable Y. Here $Y = g(X)$ for some known function g, and X is a standard normal random variable, i.e., its probability density function is $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Thus, the mean of Y may be written as

$$\mu = \int_{-\infty}^{\infty} g(x)\phi(x)\, dx.$$

One *unbiased* estimator of μ, depending on the sample size, $n = 1, 2, \ldots$ is the Simple Monte Carlo estimator:

$$\hat{Y}_{n,MC} = \frac{1}{n}\left[g(X_1) + \cdots + g(X_n)\right], \text{ where } X_i \text{ i.i.d. } N(0,1).$$

Another *unbiased* estimator of μ, also depending on the sample size, is the Importance Sampling estimator:

$$\hat{Y}_{n,IS} = \frac{1}{n}\left[h(Z_1) + \cdots + h(Z_n)\right],$$

where Z_i i.i.d. with probability density function $f_Z(z) = \frac{1}{2} e^{-|z|}$.

a) What is $h(z)$ in terms of $g(z)$?

Answer: The integral defining the mean may be written equivalently as

$$\mu = \int_{-\infty}^{\infty} g(x)\phi(x)\, dx = \int_{-\infty}^{\infty} g(z)\frac{\phi(z)}{f_Z(z)} f_Z(z)\, dz,$$

and so

$$h(z) = g(z)\frac{\phi(z)}{f_Z(z)} = \sqrt{\frac{2}{\pi}}\, g(z) e^{-z^2/2+|z|}.$$

b) Let $W = h(Z)$. Suppose that $\text{var}(Y) = \sigma^2$, and $\text{var}(W) = c\sigma^2$ for some constants σ and c. What are the root mean square errors of the Simple Monte Carlo and Importance Sampling estimators in terms of σ, c, and n?

Answer: Since the $Y_i = g(X_i)$ *are i.i.d. with common variance* σ^2*, it follows that* $\text{var}\left(\hat{Y}_{n,MC}\right) = \sigma^2/n$*, and the root mean square error for the Simple Monte Carlo is* $\sigma/\sqrt{n}$*. Similarly, since the* $W_i = h(Z_i)$ *are i.i.d. with common variance* $c\sigma^2$*, it follows that* $\text{var}\left(\hat{Y}_{n,IS}\right) = c\sigma^2/n$*, and the root mean square error for the Importance Sampling is* $\sigma\sqrt{c/n}$*.*

c) For $n = 100$, you compute both a Simple Monte Carlo and an Importance Sampling estimate. The sample variance of the Y_i is 0.250 and the sample variance of the W_i is 0.0625. What do you estimate c to be?

Answer:

$$c = \frac{\text{var}(W)}{\text{var}(Y)} \approx \frac{\widehat{\text{var}}(W)}{\widehat{\text{var}}(Y)} = \frac{0.0625}{0.250} = \frac{1}{4}.$$

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d) How large should n be to obtain confidence intervals of half-width 0.001 for these two estimators? If the time required to compute these estimators is $0.0005n$ seconds, then how much time is saved by using Importance Sampling over Simple Monte Carlo?

Answer: The half-width of the confidence interval of an estimator $\hat{Y}_n$ based on n i.i.d. samples Y_i is $1.96\sqrt{\widehat{\mathrm{var}}(Y_1)/n}$, so we want for Simple Monte Carlo

$$0.001 \geq 1.96\sqrt{\frac{0.250}{n}} \implies n = 0.250\left(\frac{1.96}{0.001}\right)^2 = 960400$$

and for Importance Sampling

$$0.001 \geq 1.96\sqrt{\frac{0.0625}{n}} \implies n = 0.0625\left(\frac{1.96}{0.001}\right)^2 = 240100$$

Importance Sampling takes 1/4 the number of operations and 1/4 the time: about two minutes rather than eight minutes for Simple Monte Carlo.

2. (12 points)
A zero-inflated Poisson random variable, Y, describes how many crashes of computers in a lab occur in a month. This random variable takes on non-negative integer values and has a probability mass function

$$f_Y(y) = \Pr(Y = y) = \begin{cases} 0.4, & y = 0, \\ \frac{0.6}{(e-1)y!}, & y = 1, 2, \ldots. \end{cases}$$

a) Using the uniform pseudorandom numbers above, compute *five i.i.d. samples*, $Y_1, \ldots, Y_5$, where the Y_i come from the distribution described above.

Answer: The cumulative distribution function of Y is given by $F_Y(y) = f_Y(0)+\cdots+f_Y(y)$, or in tabular form:

y	0	1	2	3	$\cdots$
$f_Y(y)$	0.4000	0.3492	0.1746	0.0582	$\cdots$
$F_Y(y)$	0.4000	0.7492	0.9238	0.9820	$\cdots$

This translates into an inverse cumulative probability distribution given by

x	$0 \leq x \leq 0.4000$	$0.4000 < x \leq 0.7492$	$0.7492 < x \leq 0.9238$	$0.9238 < x \leq 0.9820$
$F_Y^{-1}(x)$	0	1	2	3

Reading from this table, the inverse cumulative distribution function of the first five pseudorandom numbers are

$$F_Y^{-1}(0.1622) = 0, \quad F_Y^{-1}(0.7943) = 2, \quad F_Y^{-1}(0.3112) = 0,$$
$$F_Y^{-1}(0.5285) = 1, \quad F_Y^{-1}(0.1656) = 0.$$

b) Using the uniform pseudorandom numbers above, compute *five stratified samples* with five strata and one sample per stratum, $Y_1, \ldots, Y_5$, where the Y_i come from the distribution described above.

Answer: First we generate a uniform stratified sample:

$$z_1 = \frac{0 + 0.1622}{5} = 0.0324, \quad z_2 = \frac{1 + 0.7943}{5} = 0.3589, \quad z_3 = \frac{2 + 0.3112}{5} = 0.4622,$$
$$z_4 = \frac{3 + 0.5285}{5} = 0.7057, \quad z_5 = \frac{4 + 0.1656}{5} = 0.8331.$$

The inverse cumulative distribution function of the first five pseudorandom numbers are

$$F_Y^{-1}(0.0324) = 0, \quad F_Y^{-1}(0.3589) = 0, \quad F_Y^{-1}(0.4622) = 1,$$
$$F_Y^{-1}(0.7057) = 1, \quad F_Y^{-1}(0.8331) = 2.$$

c) The true mean number of computer crashes per month from this zero-inflated Poisson distribution is 0.9492. Estimate the mean number of crashes per month based on the simple random sample in part a) and based on the stratified sample in part b). Which of these two estimates is closer to the true answer?

Answer:

$$\hat{\mu}_{MC} = \frac{1}{5}(0 + 2 + 0 + 1 + 0) = 0.6, \qquad \hat{\mu}_{SS} = \frac{1}{5}(0 + 0 + 1 + 1 + 2) = 0.8,$$

so the stratified sampling estimate is more accurate.

3. (14 points)
The following table shows four possible sets of 8 points: two based on the pseudorandom numbers at the beginning of the exam, one 4×2 grid, and one rank-1 lattice:

i	Pseudorandom $\boldsymbol{x}_i$	Pseudorandom $\boldsymbol{x}_i$	Grid $\boldsymbol{x}_i$	Rank-1 Lattice $\boldsymbol{x}_i$
1	$(0.1622, 0.7943)$	$(0.1622, 0.6892)$	$(1/8, 1/4)$	$(1/16, 1/16)$
2	$(0.3112, 0.5285)$	$(0.7943, 0.7482)$	$(3/8, 1/4)$	$(3/16, 7/16)$
3	$(0.1656, 0.6020)$	$(0.3112, 0.4505)$	$(5/8, 1/4)$	$(5/16, 13/16)$
4	$(0.2630, 0.6541)$	$(0.5285, 0.0838)$	$(7/8, 1/4)$	$(7/16, 3/16)$
5	$(0.6892, 0.7482)$	$(0.1656, 0.2290)$	$(1/8, 3/4)$	$(9/16, 9/16)$
6	$(0.4505, 0.0838)$	$(0.6020, 0.9133)$	$(3/8, 3/4)$	$(11/16, 15/16)$
7	$(0.2290, 0.9133)$	$(0.2630, 0.1524)$	$5/8, 3/4)$	$(13/16, 5/16)$
8	$(0.1524, 0.8258)$	$(0.6541, 0.8258)$	$(7/8, 3/4)$	$(15/16, 11/16)$

a) Note that the two sets of 8 two-dimensional pseudorandom points are constructed from the 16 pseudorandom points at the beginning of the exam but using different orderings. Which of these two pseudorandom sets is a more valid construction of numbers to emulate i.i.d. uniform samples $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_8$, or are they both equally valid? Explain your answer.

Answer: Because the pseudorandom numbers are meant to emulate i.i.d. uniform random numbers, combining them in either order to obtain the two-dimensional vectors is equally valid.

b) The rank-1 integration lattice is a deterministic set of points meant to be evenly spread over the unit square. Explain how the points in the rank-1 lattice are more evenly spread than pseudorandom or grid points.

Answer: The rank-1 lattice has eight different evenly spaced values for each coordinate. This is not the case for a grid. Also the points do not leave as much empty spaces between them like the pseudorandom points do.

c) Consider the three estimators of the double integral $\mu = \int_0^1 \int_0^1 g(\boldsymbol{x})\,\mathrm{d}x_1\mathrm{d}x_2$ that take the following form:

$$\hat{\mu} = \frac{1}{8}\sum_{i=1}^{8} g(\boldsymbol{z}_i),$$

$$\text{where } \begin{cases} \text{i) the } \boldsymbol{z}_i \text{ are i.i.d. uniform random vectors in } [0,1]^2 \\ \text{ii) the } \boldsymbol{z}_i \text{ are the points in the rank-1 integration lattice above, or} \\ \text{iii) } \boldsymbol{z}_i = \boldsymbol{x}_i + \boldsymbol{\Delta} \pmod 1, \text{ the } \boldsymbol{x}_i \text{ are the points in the rank-1 integration lattice} \\ \qquad \text{above and } \boldsymbol{\Delta} \text{ is a uniform random vector in } [0,1]^2. \end{cases}$$

Which one or more of these three estimators are unbiased? Which one or more of these estimators have zero variance?

Answer: The first and third estimators are unbiased because $E[g(\boldsymbol{z}_i)] = \mu$ in these two cases. The second estimator has zero variance because it is deterministic.

d) The rank-1 integration lattice is a deterministic set of points meant to be evenly spread over the unit square. If the coordinates of the points in the rank-1 integration lattice are rearranged as follows, are the points still evenly spread? Explain why or why not.

i	Rearranged Point Coordinates Rank-1 Lattice $\boldsymbol{x}_i$
1	$(1/16, 9/16)$
2	$(1/16, 9/16)$
3	$(3/16, 11/16)$
4	$(7/16, 15/16)$
5	$(5/16, 13/16)$
6	$(13/16, 5/16)$
7	$(7/16, 15/16)$
8	$(3/16, 11/16)$

Answer: No, the coordinates of the points of the rank-1 lattice are not i.i.d. random numbers. The rearranged net does not have evenly spread points. Also, some points are repeated for this rearranged lattice.

4. (12 points)
Consider a discretely monitored down and out put option that has a life of $T = 1$ year, and is monitored at times $1/2$ year and 1 year. This option pays off if the stock price at $T = 1$ lies below the strike price but the stock price has not fallen below the barrier at times $t = 1/2$ or 1. Assume the discrete time geometric Brownian motion model for the stock price (1) with $S(0) = 100$, $r = 3\%$, $\sigma = 70\%$, a barrier price for the option of 60 and a strike price of $K = 100$.

a) Using the pseudorandom numbers above, compute *one* stock path.

Answer: Here $d = 2$, *and so* $T/d = 1/2$. *We use the uniform* $\boldsymbol{x}_i$ *above and then compute* $z_{ij} = \Phi^{-1}(x_{ij})$ *using the table above. Thus,*

$$S(1/2) = 100 \exp\left((0.03 - 0.7^2/2)(1/2) + 0.7\sqrt{1/2}(-0.9855)\right) = 55.14,$$
$$S(1) = 55.14 \exp\left((0.03 - 0.7^2/2)(1/2) + 0.7\sqrt{1/2}(0.8214)\right) = 74.36.$$

b) Below are seven more stock paths.

Path	$S(0)$	$S(1/2)$	$S(1)$
2	100	70.38	65.48
3	100	55.52	56.66
4	100	114.66	143.37
5	100	62.19	109.59
6	100	94.19	316.15
7	100	44.52	37.23
8	100	48.50	104.77

Which of these eight paths (the one from part a) and the seven above) have a positive payoff?

Answer: Paths 2, 4, 5, *and* 6 *stay above the barrier, and of these only path* 2 *is in the money at* $t = 1$.

c) What is the estimated price for this option based on these eight paths?

Answer: The average discounted payoff for the 8 *paths is*

$$\frac{1}{8}(100 - 65.48)e^{-0.03} = 4.19,$$

which corresponds to the estimated option price.

d) Using the first point of the rank-1 lattice above in the beginning of question 3, compute *one* stock path. What is the payoff for this stock path?

Answer: Again $d = 2$, *and we compute* $z_{ij} = \Phi^{-1}(x_{ij})$, *but now for the rank-1 lattice. Thus,*

$$S(1/2) = 100 \exp\left((0.03 - 0.7^2/2)(1/2) + 0.7\sqrt{1/2}(-1.534)\right) = 42.03,$$
$$S(1) = 42.03 \exp\left((0.03 - 0.7^2/2)(1/2) + 0.7\sqrt{1/2}(-1.534)\right) = 17.67.$$

This path falls below the barrier and does not pay off.