MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell **Fall 2009**

In-Class Part of Final Exam **Wednesday, December 9**

Instructions:

i. *This exam consists of FOUR questions for a total of 50 points possible. Answer all of them.*

ii. *The time allowed for this exam is 120 minutes*

iii. *This exam is closed book, but you may use* four *double-sided letter-size sheets of notes.*

iv. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (12 points)
Consider the problem of estimating, μ, the average weekly load on a power grid. Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables giving the daily load for weekdays (Monday through Friday), and $Y_1, Y_2, \ldots$ be i.i.d. random variables giving the daily load for weekends (Saturday and Sunday). Assume that the X_i and the Y_i are independent of each other. Suppose

$$E[X_i] = \mu_X, \quad E[Y_i] = \mu_Y, \quad \text{var}(X_i) = \sigma^2, \quad \text{var}(Y_i) = \sigma^2/4.$$

Let

$$\bar{X} = \frac{1}{n_X}(X_1 + \cdots + X_{n_X})$$

be the sample mean of n_X values of the X_i, and let

$$\bar{Y} = \frac{1}{n_Y}(Y_1 + \cdots + Y_{n_Y})$$

be the sample mean of n_Y values of the Y_i.

a) Show that $\hat{\mu} = (5/7)\bar{X} + (2/7)\bar{Y}$ is an unbiased estimate of the average weekly load, μ.

Answer: Since there are five weekdays and two weekend days in a week, it follows that $\mu = (5/7)\mu_X + (2/7)\mu_Y$. *Furthermore,*

$$E(\hat{\mu}) = E[(5/7)\bar{X} + (2/7)\bar{Y}] = (5/7)\mu_X + (2/7)\mu_Y = \mu,.$$

so $\hat{\mu}$ *is unbiased.*

b) What is the variance of the estimator $\hat{\mu}$?

Answer:

$$\begin{aligned}\text{var}(\hat{\mu}) &= \text{var}[(5/7)\bar{X} + (2/7)\bar{Y}] = (5/7)^2\,\text{var}(\bar{X}) + (2/7)^2\,\text{var}(\bar{Y}) \\ &= \frac{25\sigma^2}{49n_X} + \frac{4\sigma^2/4}{49n_Y} = \frac{\sigma^2}{49}\left(\frac{25}{n_X} + \frac{1}{n_Y}\right).\end{aligned}$$

c) For a budget of $n_X + n_Y = 10000$ samples, what choice of n_X and n_Y gives the minimum variance for the estimator $\hat{\mu}$?

Answer: One wants to minimize

$$\mathrm{var}(\hat{\mu}) = \frac{\sigma^2}{49}\left(\frac{25}{n_X} + \frac{1}{10000 - n_X}\right).$$

This may be done by taking the derivative with respect to n_X *and setting it to zero, which yields:*

$$\begin{aligned}
0 &= \frac{-25}{n_X^2} + \frac{1}{(10000 - n_X)^2}, \\
\frac{25}{n_X^2} &= \frac{1}{(10000 - n_X)^2}, \\
25(10000 - n_X)^2 &= n_X^2, \\
5(10000 - n_X) &= n_X, \\
n_X &= \frac{50000}{6} \approx 8333, \qquad n_Y = \frac{10000}{6} \approx 1667, \\
\mathrm{var}(\hat{\mu}) &= \frac{\sigma^2}{49}\left(\frac{25 \times 6}{50000} + \frac{6}{10000}\right) = \frac{6\sigma^2}{49 \times 10000}(5+1) = \frac{9\sigma^2}{122500}.
\end{aligned}$$

For the next three problems you will need the following *uniform* pseudorandom numbers:

$$0.81472, 0.15761, 0.65574, 0.70605, 0.43874, 0.27603, 0.75127, 0.84072, 0.35166, 0.07585, \ldots$$

---

2. (12 points)
Consider the problem of approximating the integral

$$\int_1^2 \frac{e^{-x}}{x^2}\,\mathrm{d}x.$$

a) Use the *first four uniform pseudorandom numbers* above to approximate this integral.

Answer: We take the pseudorandom numbers above and add 1 *to put them in the interval for integration.*

$$\int_1^2 \frac{e^{-x}}{x^2}\,\mathrm{d}x \approx \frac{1}{4}\left(\frac{e^{-1.81472}}{1.81472^2} + \frac{e^{-1.15761}}{1.15761^2} + \frac{e^{-1.65574}}{1.65574^2} + \frac{e^{-1.70605}}{1.70605^2}\right) = 0.10400$$

b) Use *a total of four antithetic variates* to approximate this integral.

Answer: Note that $2 - 0.81472 = 1.18528$ *and* $2 - 0.15761 = 1.84239$, *so*

$$\int_1^2 \frac{e^{-x}}{x^2}\,\mathrm{d}x \approx \frac{1}{4}\left(\frac{e^{-1.81472}}{1.81472^2} + \frac{e^{-1.15761}}{1.15761^2} + \frac{e^{-1.18528}}{1.18528^2} + \frac{e^{-1.84239}}{1.84239^2}\right) = 0.13705.$$

By the way, the numerical approximation using MATLAB's quad function is 0.12973, *so the antithetic variates give a more accurate answer in this case.*

3. (12 points)
The Pareto distribution has a probability density function defined by

$$f(x) = \frac{1}{x^2}, \quad 1 \le x < \infty.$$

a) Use the *uniform* pseudorandom numbers above to compute *four Pareto pseudorandom numbers.*

Answer: The cumulative distribution function is $F(x) = \int_1^x f(t)\,\mathrm{d}t = 1 - 1/x$. *The inverse cumulative probability distribution is given by setting* $u = F(x) = 1 - 1/x$ *and solving for* x, *i.e.,* $x = 1/(1-u)$. *Thus, we have*

u_i	0.81472	0.15761	0.65574	0.70605
x_i	5.39734	1.18710	2.90479	3.40189

b) Use these four Pareto pseudorandom numbers to estimate the integral

$$\int_1^\infty \frac{e^{-x}}{x^2}\,\mathrm{d}x.$$

Answer:

$$\int_1^\infty \frac{e^{-x}}{x^2}\,\mathrm{d}x \approx \frac{1}{4}\left(e^{-5.39734} + e^{-1.18710} + e^{-2.90479} + e^{-3.40189}\right) = 0.09943.$$

The true answer is ≈ 0.15, *so the small sample size hurts the accuracy.*

4. (14 points)
Consider the following 8-point, 2-dimensional unshifted rank-1 lattice $\{\boldsymbol{x}_i\}_{i=1}^8$:

i	Unshifted lattice $\boldsymbol{x}_i$
1	$(0.000, 0.000)$
2	$(0.125, 0.375)$
3	$(0.250, 0.750)$
4	$(0.375, 0.125)$
5	$(0.500, 0.500)$
6	$(0.625, 0.875)$
7	$(0.750, 0.750)$
8	$(0.875, 0.625)$

For $r = 1, \ldots, 30$, let $\{\boldsymbol{t}_i^{(r)}\}_{i=1}^{8}$ be shifted copies of this rank-1 lattice where

$$\boldsymbol{t}_i^{(r)} = \boldsymbol{x}_i + \boldsymbol{\Delta}^{(r)} \pmod 1,$$

and $\boldsymbol{\Delta}^{(1)}, \ldots, \boldsymbol{\Delta}^{(30)} \in [0,1)^2$ are i.i.d. uniform 2-dimensional vectors. This problem concerns the approximation of

$$\mu = \int_{[0,1]^2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

for some function $f : [0,1]^2 \to \mathbb{R}$. Let

$$\hat{\mu}_r = \frac{1}{8}\sum_{i=1}^{8} f(\boldsymbol{t}_1^{(r)}), \quad r = 1, \ldots, 30, \qquad \hat{\mu} = \frac{1}{30}\sum_{r=1}^{30} \hat{\mu}_r.$$

a) Are $\boldsymbol{t}_1^{(1)}$ and $\boldsymbol{t}_8^{(1)}$ independent?

Answer: Note that

$$\begin{aligned}
\boldsymbol{t}_8^{(1)} - \boldsymbol{t}_1^{(1)} \bmod 1 &= (\boldsymbol{x}_8 + \boldsymbol{\Delta}^{(1)} \bmod 1) - (\boldsymbol{x}_1 + \boldsymbol{\Delta}^{(1)} \bmod 1) \\
&= \boldsymbol{x}_8 - \boldsymbol{x}_1 + (\boldsymbol{\Delta}^{(1)} - \boldsymbol{\Delta}^{(1)}) \quad \bmod 1 \\
&= (0.875, 0.625) \quad \bmod 1
\end{aligned}$$

Thus, $\boldsymbol{t}_8^{(1)} = (0.875, 0.625) + \boldsymbol{t}_1^{(1)} \pmod 1$. *Given* $\boldsymbol{t}_1^{(1)}$, *one knows* $\boldsymbol{t}_8^{(1)}$ *exactly. They are dependent.*

b) Are $\boldsymbol{t}_1^{(1)}$ and $\boldsymbol{t}_8^{(2)}$ independent?

Answer: Since

$$\begin{aligned}
\boldsymbol{t}_8^{(2)} &= (0.875, 0.625) + \boldsymbol{t}_1^{(1)} + (\boldsymbol{\Delta}^{(2)} - \boldsymbol{\Delta}^{(1)}) \pmod 1 \\
&= (0.875, 0.625) + \boldsymbol{t}_1^{(1)} + \boldsymbol{\Delta} \pmod 1
\end{aligned}$$

for some $\boldsymbol{\Delta}$ *uniform on* $[0,1)^2$, *it follows that* $\boldsymbol{t}_1^{(1)}$ *and* $\boldsymbol{t}_8^{(2)}$ *are independent.*

c) Suppose that $\hat{\mu} = 1.56$, $\operatorname{var}(f(\boldsymbol{t}_i^{(r)})) \approx 4$ and $\operatorname{var}(\hat{\mu}_r) \approx 1/16$. Give an approximate 95% confidence interval for μ.

Answer: Since $\operatorname{var}(\hat{\mu}) = \operatorname{var}(\hat{\mu}_r)/30 \approx 1/480$, *it follows that the confidence interval is*

$$1.56 \pm 1.96\sqrt{\frac{1}{480}} = 1.56 \pm 0.089 = [1.47, 1.65].$$

d) Using the pseudo-random numbers above, compute $\boldsymbol{t}_1^{(1)}$ and $\boldsymbol{t}_8^{(2)}$.

Answer:

$$\begin{aligned}
\boldsymbol{t}_1^{(1)} &= (0.000, 0.000) + (0.81472, 0.15761) \pmod 1 = (0.81472, 0.15761) \\
\boldsymbol{t}_8^{(2)} &= (0.875, 0.625) + (0.65574, 0.70605) \pmod 1 = (0.53074, 0.33105)
\end{aligned}$$