MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell **In-Class Final Exam** **8-10 AM, Tuesday, December 3, 2013**

Instructions:

i. *This in-class part of the final exam has FOUR questions for a total of* 65 *points possible. You should attempt them all.*

ii. *The time allowed is 120 minutes.*

iii. *Unless otherwise indicated, give answers to at least four significant digits.*

iv. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

v. *(Programmable) calculators are allowed, but they must not have stored text.*

vi. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (10 points)
A computer simulation of a call center records the number of customers having to wait more than 10 minutes for an associate. When simulating 1000 days of calls it is found that the sample average number of customers per day waiting more than 10 minutes is 53.45 with a sample standard deviation of 15.31. Use the Central Limit Theorem to construct an approximate 99% confidence interval for the population average number of customers per day needing to wait more than 10 minutes.

Answer: The average number of customers having to wait for 10 *minutes or more is*

$$\hat{\mu} \pm \frac{2.58 \times 1.1 \times \hat{\sigma}}{\sqrt{n}} = 53.45 \pm \frac{2.58 \times 1.1 \times 15.31}{\sqrt{1000}} = 53.45 \pm 1.374 = [52.08, 54.82]$$

2. (10 points)
A simulation of the price of an Asian Arithmetic Mean Option using $d = 256$ time steps and $n = 10^5$ sample paths costs 4 seconds of computer time.

a) The error (half-width of the 99% confidence interval) is 0.05, and the required error tolerance is 0.02. How many samples should be used to achieve the required error tolerance? How much computer time should it take to achieve the required tolerance?

Answer: Since the error is proportional to the square root of the number of samples, the number of samples needed is $(0.05/0.02)^2 \times 10^5 = 6.25 \times 10^5$. *The time taken will be* $4 \times 6.25 = 25$ *seconds.*

b) Estimate the seconds of computer time required if only $d = 16$ time steps and 10^5 samples are used.

Answer: Since the number of computer operations required is roughly proportional to the number of time steps used, the time for 16 *time steps is roughly* $4/16 = 0.25$ *seconds.*

3. (22 points)
Again consider pricing an Asian Arithmetic Mean Option. Let $Y^{(d)}$ denote the payoff of the option based on d time steps, and $\mu_d = \mathbb{E}[Y^{(d)}]$. You want to compute μ_{256} as in the previous problem. The estimator mentioned in the previous problem is

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i^{(256)}, \qquad Y_i^{(256)} \text{ are IID instances of } Y^{(256)}.$$

A *multi-level* estimator takes the form

$$\hat{\mu}_{n,\mathrm{ML}} = \frac{1}{n}\sum_{i=1}^{n} \widetilde{Y}_i, \qquad \widetilde{Y}_i \text{ are IID instances of } \widetilde{Y},$$

$$\widetilde{Y} = \begin{cases} \dfrac{4Y^{(16)}}{3} & \text{with probability } 3/4, \\ 4[Y^{(256)} - Y^{(16)}] & \text{with probability } 1/4. \end{cases}$$

a) Show that $\hat{\mu}_{n,\mathrm{ML}}$ is an *unbiased* estimator of μ_{256}.

Answer:

$$\mathbb{E}(\hat{\mu}_{n,ML}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(\widetilde{Y}_i) = \mathbb{E}(\widetilde{Y}) = \frac{4\mathbb{E}[Y^{(16)}]}{3} \times \frac{3}{4} + 4\mathbb{E}[Y^{(256)} - Y^{(16)}] \times \frac{1}{4}$$
$$= \mathbb{E}[Y^{(256)}] = \mu_{256}.$$

b) Suppose that

$$\mu_{16} = 5v, \quad \mu_{256} = 4v,$$
$$\mathbb{E}[(Y^{(16)})^2] = 60v^2, \quad \mathbb{E}[(Y^{(256)})^2] = 56v^2, \quad \mathbb{E}[(Y^{(256)} - Y^{(16)})^2] = 2v^2.$$

Derive the *variance* of $\hat{\mu}_n$ and $\hat{\mu}_{n,\mathrm{ML}}$ in terms of v and n.

Answer:

$$\mathrm{var}(\hat{\mu}_n) = \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i^{(256)}\right) = \frac{\mathrm{var}(Y_i^{(256)})}{n} = \frac{\mathbb{E}[(Y_i^{(256)})^2] - \mu_{256}^2}{n}$$
$$= \frac{56v^2 - 16v^2}{n} = \frac{40v^2}{n}$$

$$\mathrm{var}(\hat{\mu}_{n,ML}) = \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} \widetilde{Y}_i\right) = \frac{\mathrm{var}(\widetilde{Y})}{n}$$

$$\mathbb{E}[\widetilde{Y}^2] = \frac{16\mathbb{E}[(Y^{(16)})^2]}{9} \times \frac{3}{4} + 16\mathbb{E}[(Y^{(256)} - Y^{(16)})^2] \times \frac{1}{4}$$
$$= \frac{4(60v^2)}{3} + 4(2v^2) = 88v^2$$

$$\mathrm{var}(\hat{\mu}_{n,ML}) = \frac{\mathbb{E}[\widetilde{Y}^2] - [\mathbb{E}(\widetilde{Y})]^2}{n} = \frac{88v^2 - (4v)^2}{n} = \frac{72v^2}{n}$$

c) Suppose that the *cost* to generate $Y^{(16)}$ is c, the cost to generate $Y^{(256)}$ is $16c$, and the cost to generate $Y^{(256)} - Y^{(16)}$ is $17c$. Compute the expected cost of $\hat{\mu}_n$ and the expected cost of $\hat{\mu}_{n,\mathrm{ML}}$ in terms of c and n.

Answer:

$$\mathrm{cost}(\hat{\mu}_n) = n\,\mathrm{cost}(Y^{(256)}) = 16cn,$$
$$\mathrm{cost}(\hat{\mu}_{n,ML}) = n\left[\mathrm{cost}(Y^{(16)})\frac{3}{4} + \mathrm{cost}(Y^{(256)} - Y^{(16)})\frac{1}{4}\right] = n\frac{3c+17c}{4} = \frac{20cn}{4} = 5cn.$$

d) Using the results in the previous parts, if the expected cost budget is N, what are the mean square errors $\hat{\mu}_n$ and $\hat{\mu}_{n,\mathrm{ML}}$? For what values of v and c does the multi-level method have a smaller mean square error?

Answer: For $\hat{\mu}_n$, we have $N = 16cn$, and so $n = N/(16c)$. Thus,

$$\mathrm{MSE}(\hat{\mu}_n) = \mathrm{var}(\hat{\mu}_n) = \frac{40v^2}{n} = \frac{40v^2 \times 16c}{N} = \frac{640v^2c}{N}.$$

For $\hat{\mu}_{n,ML}$, we have $N = 5cn$, and so $n = N/(5c)$. Thus,

$$\mathrm{MSE}(\hat{\mu}_{n,ML}) = \mathrm{var}(\hat{\mu}_{n,ML}) = \frac{72v^2}{n} = \frac{72v^2 \times 5c}{N} = \frac{360v^2c}{N} < \frac{640v^2c}{N}.$$

So the multi-level method has a smaller error for all values of v and c.

4. (23 points)
Consider the problem of pricing a kind of basket call option with two stocks:

$$S_1(t) = \$100\mathrm{e}^{(r-\sigma_1^2/2)t+\sigma_1 B_1(t)}, \qquad S_2(t) = \$100\mathrm{e}^{(r-\sigma_2^2/2)t+\sigma_2[0.6B_1(t)+0.8B_2(t)]},$$
$$\text{discounted payoff} = \max([S_1(T/2) + S_1(T) + S_2(T/2) + S_2(T)]/4 - K, 0)\mathrm{e}^{-rT},$$
$$r = 1\% \text{ per year}, \quad \sigma_1 = 40\% \text{ per } \sqrt{\text{year}}, \quad \sigma_2 = 60\% \text{ per } \sqrt{\text{year}}, \quad T = 1/2 \text{ year}, \quad K = \$100.$$

Here B_1 and B_2 are independent Brownian motions. Use the pseudo-random IID $\mathcal{N}(0,1)$ numbers

$$1.4234, 1.1188, 0.7863, -1.2447, -1.0660, 0.9469, \ldots$$

to compute *one* discounted payoff based on one path each of the two stocks.

Answer: The two Brownian motions are computed as

$$\begin{aligned} B_1(0) &= 0 & B_2(0) &= 0,\\ B_1(1/4) &= \sqrt{1/4}X_{11}, & B_2(1/4) &= \sqrt{1/4}X_{21}\\ B_1(1/2) &= B_1(1/4) + \sqrt{1/4}X_{12}, & B_2(1/2) &= B_2(1/4) + \sqrt{1/4}X_{22}. \end{aligned}$$

with

X_{11}	X_{12}	X_{21}	X_{22}
1.4234	1.1188	0.7863	−1.2447

If you compute $B_1(T/2), B_1(T), B_2(T/2), B_2(T)$, then you get:

t	$1/4$	$1/2$
$B_1(t)$	0.7117	1.2711
$B_2(t)$	0.3932	−0.2292
$S_1(t)$	130.63	160.55
$S_2(t)$	149.54	130.03

and the discounted payoff is

$$\max([S_1(1/4) + S_1(1/2) + S_2(1/4) + S_2(1/2)]/4 - 100, 0)\mathrm{e}^{-r/2} = \$42.47.$$

If you compute $B_1(T/2), B_2(T/2), B_1(T), B_2(T)$, then you get:

t	$1/4$	$1/2$
$B_1(t)$	0.7117	1.1048
$B_2(t)$	0.5594	−0.0629
$S_1(t)$	130.63	150.22
$S_2(t)$	161.97	132.65

and the discounted payoff is

$$\max([S_1(1/4) + S_1(1/2) + S_2(1/4) + S_2(1/2)]/4 - 100, 0)\mathrm{e}^{-r/2} = \$43.65.$$