MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell **In-Class Final Exam** **Tuesday, December 9, 2014**

Instructions:

i. *This in-class part of the final exam has FOUR questions for a total of* 65 *points possible. You should attempt them all.*

ii. *The time allowed is 120 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

vi. *Off-site students may contact the instructor at 630-696-8124.*

1. (15 points)
   Consider a sequence of IID random samples, $Y_1, Y_2, \ldots$.

   a) If the sample mean and variance of the first $n = 10^4$ random variables are $\hat{\mu}_{10000} = 47.29$ and $s^2_{10000} = 13.56$, respectively, construct an approximate Central Limit Theorem 99% confidence interval for the true (population) mean of Y.

      Answer:
      $$\hat{\mu}_{1000} \pm \frac{2.58 s_{10000}}{\sqrt{10000}} = 47.29 \pm 0.0949 = [47.20, 47.38]$$

   b) For the situation in part a), how large a sample size would be required to make the half-width of the confidence interval no greater than 0.01?

      Answer:
      $$0.01 \ge \frac{2.58 s_{10000}}{\sqrt{n}} \implies n \ge \lceil 258^2 \times 13.56 \rceil \approx 9.00 \times 10^5$$

   c) Suppose that 4123 of the first 10000 Y_i are at least as large as 50. Based on the Central Limit Theorem, construct an approximate 99% confidence interval for $\Pr(Y \ge 50)$.

      Answer: The sample proportion is $\hat{p}_{10000} = 0.4123$*, and so an approximate CLT confidence interval is*
      $$\hat{p}_{10000} \pm \frac{2.58\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{10000}} = 0.4123 \pm 0.0127 = [0.3996, 0.4250]$$

2. (10 points)
   Let $f : \mathbb{R} \to \mathbb{R}$ be some function whose integral you wish to compute with respect to a non-negative weight function, w, i.e.,
   $$\mu = \int_{-\infty}^{\infty} f(x)\, w(x)\, \mathrm{d}x = ?$$

Let $W(x) := \int_{-\infty}^{x} w(t)\,\mathrm{d}t$, and let $C = \lim_{x\to\infty} W(x)$. Let $U_1, \ldots, U_n$ be IID $\mathcal{U}[0,1]$ random variables. Use these U_i to construct an unbiased estimate for μ.

Answer: First perform a change of variable. Let $y = W(x)$ and $x = W^{-1}(y)$. Thus, $\mathrm{d}y = W'(x)\mathrm{d}x$ and

$$\mu = \int_0^C f(W^{-1}(y))\,\mathrm{d}y.$$

Next let $y = Cu$ and $u = y/C$. Then

$$\mu = C\int_0^1 f(W^{-1}(Cu))\,\mathrm{d}u.$$

Thus,

$$\hat{\mu}_n = \frac{C}{n}\sum_{i=1}^{n} f(W^{-1}(CU_i))$$

is an unbiased estimate of μ.

3. (20 points)
Let $f : \mathbb{R}^d \to \mathbb{R}$ be some function whose integral you wish to compute with respect to a probability density function (PDF), ϱ, i.e.,

$$\mu = \int_{\mathbb{R}^d} f(\boldsymbol{x})\,\varrho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Suppose that it is difficult to generate random variables with PDF ϱ, but easy to generate random variables with PDF $\tilde{\varrho}$. Moreover, for some $c > 0$, $c\varrho(\boldsymbol{x}) \le \tilde{\varrho}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^d$.

a) Construct an estimate of μ using acceptance-rejection sampling with $\widetilde{\boldsymbol{X}}_1, \ldots, \widetilde{\boldsymbol{X}}_n$ IID $\sim \tilde{\varrho}$ and $U_1, \ldots, U_n$ IID $\mathcal{U}[0,1]$ that are also independent from the $\widetilde{\boldsymbol{X}}_i$.

Answer: The acceptance-rejection method does a loop

Let $j = 0$
For $i = 1, \ldots, n$
If $U_i \le c\varrho(\widetilde{\boldsymbol{X}}_i)/\tilde{\varrho}(\widetilde{\boldsymbol{X}}_i)$, then let $j = j + 1$ and $\boldsymbol{X}_j = \widetilde{\boldsymbol{X}}_i$.
End
Let $N = j$.
The estimate is then

$$\hat{\mu}_{\mathrm{AR},N} = \frac{1}{N}\sum_{j=1}^{N} f(\boldsymbol{X}_j).$$

b) Construct an estimate of μ using importance sampling with $\widetilde{\boldsymbol{X}}_1, \ldots, \widetilde{\boldsymbol{X}}_n$ IID $\sim \tilde{\varrho}$.

Answer: We re-write the integral as

$$\mu = \int_{\mathbb{R}^d} f(\boldsymbol{x})\,\frac{\varrho(\boldsymbol{x})}{\tilde{\varrho}(\boldsymbol{x})}\tilde{\varrho}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

So we now think of this as an integral with respect to the density $\tilde{\varrho}$ and the estimate is

$$\hat{\mu}_{\mathrm{IS},n} = \frac{1}{n}\sum_{i=1}^{n} f(\widetilde{\boldsymbol{X}}_i)\frac{\varrho(\widetilde{\boldsymbol{X}}_i)}{\tilde{\varrho}(\widetilde{\boldsymbol{X}}_i)}$$

c) What are the differences between these two estimates?

Answer: The importance sampling estimate uses all of the $\widetilde{X}_i$, whereas the acceptance-rejection method uses only some of the $\widetilde{X}_i$. We may also think of the integral as

$$\mu = \frac{1}{c}\int_{\mathbb{R}^d}\int_0^1 f(\boldsymbol{x})\,\mathbb{1}_{[0,c\varrho(\boldsymbol{x})/\tilde{\varrho}(\boldsymbol{x})]}(u)\,\tilde{\varrho}(\boldsymbol{x})\,\mathrm{d}u\,\mathrm{d}\boldsymbol{x}.$$

Applying importance sampling to this integral would yield

$$\tilde{\mu}_{\mathrm{IS},n} = \frac{1}{cn}\sum_{i=1}^{n} f(\widetilde{\boldsymbol{X}}_i)\mathbb{1}_{[0,c\varrho(\widetilde{\boldsymbol{X}}_i)/\tilde{\varrho}(\widetilde{\boldsymbol{X}}_i)]}(U_i) = \frac{1}{cn}\sum_{j=1}^{N} f(\boldsymbol{X}_j),$$

which is similar to $\hat{\mu}_{\mathrm{AR},N}$. The only difference is that the former uses $cn = \mathbb{E}(N)$ rather than N in the factor on the left.

4. (20 points)
Consider a basket European call option based on two stocks, $S^{(1)}$, and $S^{(2)}$, modeled by two independent geometric Brownian motions, $B^{(1)}$ and $B^{(2)}$, as follows

$$S^{(1)}(t) = 100\mathrm{e}^{-0.125t+0.5B^{(1)}(t)},\ \ S^{(2)}(t) = 100\mathrm{e}^{-0.08t+0.4B^{(3)}(t)},\ \ B^{(3)}(t) := 0.6B^{(1)}(t)+0.8B^{(2)}(t).$$

The interest rate is assumed to be zero. The time to expiry, T, is three months. The payoff of the option is $\max\{S^{(1)}(T) - 100, S^{(2)}(T) - 100, 0\}$.

a) What are $\mathbb{E}[B^{(3)}(t)]$ and $\mathbb{E}[B^{(3)}(t)B^{(3)}(t+s)]$ for $s \geq 0$?

Answer: Since $B^{(1)}$ and $B^{(2)}$ are independent Brownian motions, it follows that

$$\begin{aligned}
\mathbb{E}[B^{(3)}(t)] &= \mathbb{E}[0.6B^{(1)}(t) + 0.8B^{(2)}(t)] = 0.6\,\mathbb{E}[B^{(1)}(t)] + 0.8\,\mathbb{E}[B^{(2)}(t)] = 0,\\
\mathbb{E}[B^{(3)}(t)B^{(3)}(t+s)] &= \mathbb{E}[\{0.6B^{(1)}(t) + 0.8B^{(2)}(t)\}\{0.6B^{(1)}(t+s) + 0.8B^{(2)}(t+s)\}]\\
&= 0.6^2\,\mathbb{E}[B^{(1)}(t)B^{(1)}(t+s)] + (0.8)(0.6)\,\mathbb{E}[B^{(2)}(t)B^{(1)}(t+s)]\\
&\quad + (0.6)(0.8)\,\mathbb{E}[B^{(1)}(t)B^{(2)}(t+s)] + 0.8^2\,\mathbb{E}[B^{(2)}(t)B^{(2)}(t+s)]\\
&= 0.36\times t + 0.48\times 0 + 0.48\times 0 + 0.64\times t = t
\end{aligned}$$

b) Based on the following IID $\mathcal{N}(0,1)$ random numbers, generate *two* payoffs of the basket European call option:

$$0.5268, 1.1492, 0.7640, -0.6327, 1.0802, 1.8522, 0.6952, 0.0661, \ldots$$

Answer: Letting Z_i denote the points above,

$$\begin{aligned}
B_1^{(1)}(T) &= \sqrt{T} Z_1 = 0.5(0.5268) = 0.2634 \\
B_1^{(2)}(T) &= \sqrt{T} Z_2 = 0.5(1.1492) = 0.5746 \\
B_1^{(3)}(T) &= 0.6 B_1^{(1)}(T) + 0.8 B_1^{(2)}(T) = 0.6177 \\
B_2^{(1)}(T) &= \sqrt{T} Z_3 = 0.5(0.7640) = 0.3820 \\
B_2^{(2)}(T) &= \sqrt{T} Z_4 = 0.5(-0.6327) = -0.3164 \\
B_2^{(3)}(T) &= 0.6 B_2^{(1)}(T) + 0.8 B_2^{(2)}(T) = -0.0239 \\
S_1^{(1)}(T) &= 100\mathrm{e}^{-0.125T + 0.5 B_1^{(1)}(T)} = 110.57 \\
S_1^{(2)}(T) &= 100\mathrm{e}^{-0.08T + 0.4 B_1^{(3)}(T)} = 117.32 \\
S_2^{(1)}(T) &= 100\mathrm{e}^{-0.125T + 0.5 B_2^{(1)}(T)} = 125.50 \\
S_2^{(2)}(T) &= 100\mathrm{e}^{-0.08T + 0.4 B_2^{(3)}(T)} = 97.09 \\
\text{payoff}_1 &= \max\{10.57, 25.50, 0\} = 25.50 \\
\text{payoff}_2 &= \max\{17.32, -2.91, 0\} = 17.32
\end{aligned}$$

c) Below is a set of the first four scrambled and digitally shifted 4-dimensional Sobol' points:

i	x_{i1}	x_{i2}	x_{i3}	x_{i4}
	$\boldsymbol{x}_i$			
0	0.3470	0.6293	0.2813	0.2909
1	0.9035	0.0219	0.6519	0.8898
2	0.0579	0.4613	0.0862	0.5609
3	0.6927	0.8226	0.9725	0.1490

Which point(s), if any, lie in the box, $[0,1) \times [0,1/2) \times [0,1/2) \times [0,1)$? Why is that to be expected? Which point(s), if any, lie in the box, $[0,1/2) \times [0,1/2) \times [0,1/2) \times [0,1)$? Why is that to be expected?

Answer: The point $\boldsymbol{x}_2$ lies in the first box, which is $1/4$ of the points lying inside a box of volume $1/4$, as expected. The point $\boldsymbol{x}_2$ also lies in the second box, which has volume $1/8$, but it is impossible for it to contain half a point.

d) The inverse normal transformation is of these scrambled Sobol' points is

i	$\Phi^{-1}(x_{i1})$	$\Phi^{-1}(x_{i2})$	$\Phi^{-1}(x_{i3})$	$\Phi^{-1}(x_{i4})$
0	−0.3935	0.3300	−0.5789	−0.5508
1	1.3015	−2.0158	0.3905	1.2252
2	−1.5730	−0.0972	−1.3644	0.1533
3	0.5036	0.9253	1.9181	−1.0405

Compute *one* payoff of the basket European call option using these Sobol' points.

Answer:

$$
\begin{aligned}
B_1^{(1)}(T) &= \sqrt{T}\Phi^{-1}(x_{11}) = 0.5(-0.3935) = -0.1968 \\
B_1^{(2)}(T) &= \sqrt{T}\Phi^{-1}(x_{12}) = 0.5(0.3300) = 0.1650 \\
B_1^{(3)}(T) &= 0.6B_1^{(1)}(T) + 0.8B_1^{(2)}(T) = 0.0140 \\
S_1^{(1)}(T) &= 100\mathrm{e}^{-0.125T+0.5B_1^{(1)}(T)} = 87.84 \\
S_1^{(2)}(T) &= 100\mathrm{e}^{-0.08T+0.4B_1^{(3)}(T)} = 98.57 \\
\text{payoff}_1 &= \max\{-13.16, -1.43, 0\} = 0
\end{aligned}
$$