MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell **In-Class Part of Final Exam** **Thursday, December 8, 2016**

Instructions:

i. *This part of the final exam has FIVE questions. Attempt all that you wish: all will be graded. The sum of the points possible for each question is* 76 *points, but your maximum score will be* 64 *points. Added to the* 36 *point maximum on the take-home part this gives a total maximum for the final exam of* 100 *points.*

ii. *The time allowed is 120 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (16 points)
A stock is modeled by a geometric Brownian motion with an initial price of \$30, an interest rate of 1%, and a volatility of 40%. A stock price path for the first five months is given by

t	0	1/12	1/6	1/4	1/3	5/12	1/2
$S(t)$	\$30.00	\$29.62	\$30.58	\$31.37	\$33.07	\$32.07	??

a) Using the standard normal random number $Z = -0.7573$, what should $S(1/2)$ be?

Answer: The formula for a geometric Brownian motion is

$$S(t) = S(0)\exp((r - \sigma^2/2)t + \sigma B(t)),$$

where B is a Brownian motion. This means that

$$S(t) = S(t - \Delta)\exp((r - \sigma^2/2)\Delta + \sigma\sqrt{\Delta}Z),$$

where $Z \sim \mathcal{N}(0,1)$. Thus,

$$S(1/2) = S(5/12)\exp((0.01 - 0.4^2/2)(1/12) + 0.4\sqrt{1/12}(-0.7573)) = \$29.21.$$

b) If $S(1/3) = \$32.25$ instead of the value in the table above, how would this change your answer to part a)?

Answer: No change at all since the stock price at $t = 1/2$ depends only on the price at $t = 5/12$.

c) Using the stock path above, together with the $S(1/2)$ that you computed in part a), what is the discounted payoff of a *lookback call* option that expires 1/2 year from the initial time?

Answer: The discounted payoff is

$$\left[S(1/2) - \min_{0\le t\le 1/2} S(t)\right] \mathrm{e}^{-rT} = (\$29.21 - \$29.21)\mathrm{e}^{-0.01/2} = \$0.$$

d) You want to use the discounted European call option payoff with a strike price of \$30 as a control variate. You find that the European call option price is \$3.44, and that a good control variate coefficient is $\beta = 0.9576$. In using control variates to compute the price of the lookback call option, what is the value of the observation Y_{CV} that you would construct based on the stock price path above and your answers to parts a) and c)?

Answer: Let Y be the discounted lookback call option payoff of part c), and let the discounted European call payoff for the same path be

$$X = \max(S(T) - K, 0)\mathrm{e}^{-rT} = \max(\$29.21 - \$30, 0)\mathrm{e}^{-0.01/2} = \$0$$

Then

$$Y_{\mathrm{CV}} = Y + \beta(\mu_X - X) = \$0 + 0.9576(\$3.44 - \$0) = \$3.29$$

2. (16 points)
Consider the sequence of random variables $Y_1, Y_2, \ldots$, which all have the same marginal distribution with mean μ and variance σ^2. Suppose that these random variables are *dependent* with covariances

$$\mathrm{cov}(Y_i, Y_j) = \begin{cases} \mathrm{var}(Y_i) = \sigma^2, & \text{for } i = j, \\ \rho\sigma^2, & \text{for } i = j + 1 \text{ or } i = j - 1, \\ 0, & \text{otherwise.} \end{cases}$$

where $-1/2 \le \rho \le 1/2$. Let $\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ be the sample mean of the first n of these random variables.

a) Is $\hat{\mu}_n$ an unbiased estimator for μ?

Answer: Yes, since

$$\mathbb{E}(\hat{\mu}_n) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(Y_i) = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu.$$

b) What is the variance of the estimator $\hat{\mu}_n$?

Answer:

$$\mathrm{var}(\hat{\mu}_n) = \mathbb{E}[(\hat{\mu}_n - \mu)^2] = \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - \mu\right)^2\right] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu)\frac{1}{n}\sum_{j=1}^{n}(Y_j - \mu)\right]$$

$$= \mathbb{E}\left[\frac{1}{n^2}\sum_{i,j=1}^{n}(Y_i - \mu)(Y_j - \mu)\right] = \frac{1}{n^2}\sum_{i,j=1}^{n}\mathbb{E}\left[(Y_i - \mu)(Y_j - \mu)\right] =$$

In this sum of n^2 terms,

- *there are n cases where $i = j$ and $\mathbb{E}\left[(Y_i - \mu)(Y_j - \mu)\right] = \sigma^2$,*
- *there are $2(n-1)$ cases where $i = j + 1$ or $i = j - 1$ and $\mathbb{E}\left[(Y_i - \mu)(Y_j - \mu)\right] = \rho\sigma^2$, and*

- *the remaining cases have* $\mathbb{E}\left[(Y_i-\mu)(Y_j-\mu)\right]=0$.

Thus,

$$\mathrm{var}(\hat{\mu}_n)=\frac{1}{n^2}[n\sigma^2+2(n-1)\rho\sigma^2]=\frac{\sigma^2[n+2(n-1)\rho]}{n^2}$$

c) What value of ρ between $-1/2$ and $1/2$ makes the variance of $\hat{\mu}_n$ as small as possible?

Answer: The smallest value of $2(n-1)\rho$ *possible is obtained by taking* $\rho=-1/2$*. In this case* $\mathrm{var}(\hat{\mu}_n)=\sigma^2/n^2$*, which is much smaller than the uncorrelated case of* $\rho=0$*.*

3. (16 points)
For some continuous function $f:[-1,1]^2\to\mathbb{R}$, you want to compute A, the area of the set $\Omega=\{\boldsymbol{x}\in[-1,1]^2 : f(\boldsymbol{x})\geq 3\}$. Your approximation to A is to be constructed in terms of $f(\boldsymbol{X}_1),\ldots,f(\boldsymbol{X}_n)$, where $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n \overset{\text{IID}}{\sim} \mathcal{U}[-1,1]^2$. Here are some random values of $\boldsymbol{X}_i$ and the corresponding $f(\boldsymbol{X}_i)$:

i	1	2	3	4
$\boldsymbol{X}_i$	$(-0.7730, 0.4121)$	$(0.9490, -0.5067)$	$(0.4575, -0.4880)$	$(-0.2971, -0.9520)$
$f(\boldsymbol{X}_i)$	2.172	15.42	9.265	-3.75

i	5	6	7	8
$\boldsymbol{X}_i$	$(0.4152, -0.8025)$	$(0.5992, -0.3991)$	$(0.2911, 0.2817)$	$(-0.1708, -0.3556)$
$f(\boldsymbol{X}_i)$	-0.9335	8.017	0.9568	2.698

a) Based on the above sample, what would your estimate of A be?

Answer: Let $\mathbb{1}_\Omega$ *denote the characteristic function for the set* Ω*, i.e.,* $\mathbb{1}_\Omega=\begin{cases}1, & \boldsymbol{x}\in\Omega,\\ 0, & \boldsymbol{x}\notin\Omega.\end{cases}$

The area may be written as

$$A=\int_{[-1,1]^2}\mathbb{1}_\Omega(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{[-1,1]^2}4\,\mathbb{1}_\Omega(\boldsymbol{x})\underbrace{\frac{1}{4}}_{\text{uniform PDF}}\mathrm{d}\boldsymbol{x}$$
$$=\mathbb{E}\big[4\,\mathbb{1}_\Omega(\boldsymbol{X})\big],\quad \boldsymbol{X}\sim\mathcal{U}[-1,1]^2.$$

Since there are three values of f *that are at least* 3*, the estimate of the area is*

$$\widehat{A}\approx\frac{4}{8}\sum_{i=1}^{8}\mathbb{1}_\Omega(\boldsymbol{X}_i)=\frac{4\times 3}{8}=\frac{3}{2}.$$

b) If $\boldsymbol{X}_1$ was constructed from $\boldsymbol{U}_1\sim\mathcal{U}[0,1]^2$, what was $\boldsymbol{U}_1$?

Answer: The simplest way is $\boldsymbol{X}_1=2\boldsymbol{U}_1-1$*, so* $\boldsymbol{U}_1=(\boldsymbol{X}_1+1)/2=(0.1135, 0.7060)$

c) How large of an IID $\mathcal{U}[-1,1]^2$ sample would be needed to ensure that you could estimate A with an absolute error of no more than 0.02 with 99% probability?

Answer: The quantity that we want to estimate is $A = 4\,\mathbb{E}(Y)$*, where* $Y = \mathbb{1}_\Omega(\boldsymbol{X})$ *(see above), which is a Bernoulli random variable with mean* $\mu = A/4$*. Note that*

$$\mathbb{P}(Y=1) = \mathbb{P}(Y^2=1) = \mu = 1 - \mathbb{P}(Y=0) = 1 - \mathbb{P}(Y^2=0),$$
$$\mathbb{E}(Y^2) = \mu, \qquad \mathrm{var}(Y) = \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2 = \mu - \mu^2 = \mu(1-\mu)$$

Since μ *must be between* 0 *and* 1*,* $\mathrm{var}(Y) \le 1/2(1-1/2) = 1/4$*. Letting* $\hat{\mu}$ *be an IID sample mean and using the Central Limit Theorem and* $\widehat{A} = 4\hat{\mu}$*, we have*

$$\begin{aligned}99\% \approx \mathbb{P}\left[|\mu - \hat{\mu}| \le \frac{2.58\,\mathrm{std}(Y)}{\sqrt{n}}\right] &\le \mathbb{P}\left[|\mu - \hat{\mu}| \le \frac{2.58 \times (1/2)}{\sqrt{n}}\right] \\ &= \mathbb{P}\left[|4\mu - 4\hat{\mu}| \le \frac{2.58 \times 2}{\sqrt{n}}\right] = \mathbb{P}\left[\left|A - \widehat{A}\right| \le \frac{5.16}{\sqrt{n}}\right].\end{aligned}$$

Setting $5.16/\sqrt{n} = 0.02$ *gives* $n \approx 67\,000$*.*

4. (16 points)
Consider the first several elements of the nodeset of a two-dimensional rank-1 lattice sequence:

i	0	1	2	3	4	5	6	7
z_{i1}	0	1/2	1/4	3/4	??	5/8	??	??
z_{i2}	0	1/2	??	3/4	??	1/8	??	??

a) Fill in the missing information above.

Answer:

$$\begin{aligned}(5/8, 1/8) = \boldsymbol{z}_5 = \boldsymbol{z}_{101_2} &= (\boldsymbol{z}_1 + \boldsymbol{z}_4) \bmod \boldsymbol{1} = \big((1/2,1/2) + \boldsymbol{z}_4\big) \bmod \boldsymbol{1} \\ &\Longrightarrow \boldsymbol{z}_4 = (1/8, -3/8) \bmod 1 = (1/8, 5/8) \\ \boldsymbol{z}_2 &= 2\boldsymbol{z}_4 \bmod \boldsymbol{1} = (1/4, 1/4) \\ \boldsymbol{z}_6 &= \boldsymbol{z}_{110_2} = (\boldsymbol{z}_2 + \boldsymbol{z}_4) \bmod \boldsymbol{1} = (3/8, 7/8) \\ \boldsymbol{z}_7 &= \boldsymbol{z}_{111_2} = (\boldsymbol{z}_1 + \boldsymbol{z}_2 + \boldsymbol{z}_4) \bmod \boldsymbol{1} = (7/8, 3/8)\end{aligned}$$

Thus, the nodeset is

i	0	1	2	3	4	5	6	7
z_{i1}	0	1/2	1/4	3/4	1/8	5/8	3/8	7/8
z_{i2}	0	1/2	1/4	3/4	5/8	1/8	7/8	3/8

b) Is $\boldsymbol{k} = (1,3)$ part of the dual lattice for the above set of 8 points?

Answer: Yes, since $\boldsymbol{k} \cdot \boldsymbol{z}_4 \mod 1 = 1 \times 1/8 + 3 \times 5/8 \mod 1 = 2 \mod 1 = 0$.

c) Is $\hat{\mu}_n = \frac{1}{n}\sum_{i=0}^{n-1} f(\boldsymbol{z}_i)$ generally an unbiased estimator of $\mu = \int_{[0,1]^2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$? If not, how can you modify $\hat{\mu}_n$ while preserving the even distribution of the points where f is evaluated?

Answer: The estimator $\hat{\mu}_n$ as defined is biased, since the $\boldsymbol{z}_i$ are deterministic. However replacing the $\boldsymbol{z}_i$ with $\boldsymbol{x}_i = (\boldsymbol{z}_i + \boldsymbol{\Delta}) \bmod \boldsymbol{1}$, where $\boldsymbol{\Delta} \sim \mathcal{U}[0,1]^2$ makes $\hat{\mu}_n$ unbiased, since in this case $\mathbb{E}[f(\boldsymbol{x}_i)] = \mu$.

5. (12 points)

The baker's transformation $\varphi(x) = 1 - |2x - 1|$ is used to transform an integral $\int_0^1 g(t)\,\mathrm{d}t$ into the integral $\int_0^1 f(x)\,\mathrm{d}x$ by making a change of variable $t = \varphi(x)$.

a) Find f in terms of g. Note that this transformation has a discontinuous derivative, so you need to be careful at the point(s) of discontinuity.

Answer: Since

$$\varphi(x) = 1 - |2x-1| = \begin{cases} 2x, & 0 \le x \le 1/2, \\ 2 - 2x, & 1/2 < x \le 1, \end{cases} \qquad \varphi'(x) = \begin{cases} 2, & 0 \le x \le 1/2, \\ -2, & 1/2 < x \le 1, \end{cases}$$

Thus,

$$\begin{aligned}
\int_0^1 g(\varphi(x))\,\mathrm{d}x &= \int_0^{1/2} g(\varphi(x))\,\mathrm{d}x + \int_{1/2}^1 g(\varphi(x))\,\mathrm{d}x \\
&= \int_0^{1/2} g(2x)\,\mathrm{d}x + \int_{1/2}^1 g(2-2x)\,\mathrm{d}x \\
&= \underbrace{\int_0^1 g(t)\,\frac{1}{2}\mathrm{d}t}_{t=2x,\ \mathrm{d}x=\mathrm{d}t/2} + \underbrace{\int_1^0 g(t)\left(-\frac{1}{2}\right)\mathrm{d}t}_{t=2-2x,\ \mathrm{d}x=-\mathrm{d}t/2} \\
&= \frac{1}{2}\left[\int_0^1 g(t)\,\mathrm{d}t + \int_0^1 g(t)\,\mathrm{d}t\right] = \int_0^1 g(t)\,\mathrm{d}t.
\end{aligned}$$

So $f(x) = g(\varphi(x))$ is the correct transformation.

b) Assume that g is continuous. Show that f is continuous and that f is also periodic, i.e., $f(0) = f(1)$, even if g is non-periodic. Because lattice cubature is better for periodic integrands, the baker's transformation is often used to change a non-periodic integrand into a periodic one.

Answer: Since g and φ are continuous, f is automatically continuous. Since $f(0) = g(\varphi(0)) = g(0) = g(\varphi(1)) = f(1)$, f is periodic.