# MATH 565 Monte Carlo Methods in Finance

**Fred J. Hickernell** **In-Class Part of Final Exam** **Wednesday, December 6, 2017**

*Instructions:*

i. *This part of the final exam has FIVE questions with a maximum score of* 64 *points. Added to the* 36 *point maximum on the take-home part this gives a total maximum for the final exam of* 100 *points.*

ii. *The time allowed is 120 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (12 points)
You perform a Monte Carlo simulation drawing 10 000 IID random instances of $Y$, and compute the sample mean and the sample variance:

$$\overline{Y} = \frac{1}{10\,000}\sum_{i=1}^{10\,000} Y_i, \qquad S^2 = \frac{1}{9\,999}\sum_{i=1}^{10\,000}(Y_i - \overline{Y})^2.$$

Give unbiased estimators for $\mathbb{E}(Y)$, $\operatorname{var}(Y)$, and $\mathbb{E}(Y^2)$ in terms of $\overline{Y}$ and/or $S^2$.

*Answer:* $\overline{Y}$ *is an unbiased estimator for* $\mathbb{E}(Y)$. $S^2$ *is an unbiased estimator for* $\operatorname{var}(Y)$. $\frac{1}{10\,000}\sum_{i=1}^{10\,000} Y_i^2$ *is an unbiased estimator for* $\mathbb{E}(Y^2)$. *Note that*

$$\begin{aligned}
9\,999 S^2 &= \sum_{i=1}^{10\,000}(Y_i - \overline{Y})^2 = \sum_{i=1}^{10\,000}(Y_i^2 - 2Y_i\overline{Y} + \overline{Y}^2) \\
&= \left(\sum_{i=1}^{10\,000} Y_i^2\right) - 20\,000\overline{Y}^2 - 10\,000\overline{Y}^2 = \left(\sum_{i=1}^{10\,000} Y_i^2\right) + 10\,000\overline{Y}^2 \\
\frac{9\,999}{10\,000}S^2 + \overline{Y}^2 &= \frac{1}{10\,000}\left(\sum_{i=1}^{10\,000} Y_i^2\right),
\end{aligned}$$

*which is the unbiased estimator for* $\mathbb{E}(Y^2)$.

2. (15 points)
You want to construct an approximate 99% confidence interval for $\mu = \mathbb{E}(Y)$ with a half-with of 0.1.

   a) You generate 10 000 observations of $Y$ and you find that

   $$\text{the sample mean of the } Y_i = 10.42$$
   $$\text{the sample variance of the } Y_i = 348.7$$

   What sample size would you need to construct your confidence interval for $\mu$ using IID Monte Carlo?

*Answer: Using the Central Limit Theorem (CLT) approximation, the width of the approximate confidence interval is*

$$\frac{2.58 \times 1.2 \times \sqrt{348.7}}{\sqrt{n}}$$

*Making this no greater than* 0.1 *means choosing*

$$n \geq \left\lceil \left( \frac{2.58 \times 1.2 \times \sqrt{348.7}}{0.1} \right)^2 \right\rceil \approx 334\,200$$

b) Now suppose that you generate 10 000 pairs $(Y_i, X_i)$, you know that $\mathbb{E}(X) = 2$, and you find that

$$\begin{aligned} \text{the sample mean of the } Y_i &= 10.42 \\ \text{the sample variance of the } Y_i &= 348.7 \\ \text{the sample mean of the } X_i &= 4.384 \\ \text{the sample variance of the } X_i &= 434.1 \\ \text{the sample covariance of the } Y_i \text{ and the } X_i &= 347.8 \end{aligned}$$

Given this additional information, can you use a smaller sample size than in part a) to construct your confidence interval? What would that sample size be?

*Answer: We use control variates. Let* $Y_{\text{CV}} = Y + \beta(2 - X)$. *Note that* $\mathbb{E}(Y_{\text{CV}}) = \mu$ *and* $\text{var}(Y_{\text{CV}}) = \text{var}(Y) - 2\beta \,\text{cov}(Y, X) + \beta^2 \,\text{var}(X)$. *To minimize this we choose*

$$\beta = \frac{\text{cov}(Y, X)}{\text{var}(X)} \approx \frac{347.8}{434.1} = 0.8012.$$

*Then*

$$\text{var}(Y_{\text{CV}}) \approx 348.7 - 2 \times 0.8012 \times 347.8 + 0.8012^2 \times 434.1 = 70.04$$

*and the sample size needed now is*

$$n \geq \left\lceil \left( \frac{2.58 \times 1.2 \times \sqrt{70.04}}{0.1} \right)^2 \right\rceil \approx 67\,130,$$

*which is significantly smaller than without sampling* $X$.

c) If the sample covariance of the $Y_i$ and $X_i$ were as *bad* as possible in part b), what sample size would be required to construct your confidence interval?

*Answer: Since* $\text{var}(Y_{\text{CV}}) = \text{var}(Y)[1 - \text{corr}^2(Y, X)]$, *the worst situation is* $\text{cov}(Y, X) = 0$, *in which case the sample size required is the same as in part a).*

3. (12 points)
You are simulating a random variable $X$ that represents the life of a machine part in years. There is a 30% chance that the part is defective out of the box, but otherwise the part life follows an exponential distribution. The cumulative distribution function for $X$ is

$$F(x) = \begin{cases} 0, & -\infty < x < 0, \\ 1 - 0.7\mathrm{e}^{-x/5}, & 0 \le x < \infty. \end{cases}$$

Given three IID $\mathcal{U}[0,1]$ random numbers:

| $i$ | 1 | 2 | 3 |
|---|---|---|---|
| $U_i$ | 0.4914 | 0.2845 | 0.7976 |

find three IID random numbers, $X_1$, $X_2$, and $X_3$, which follow the distribution for the part life above.

*Answer: We use the inverse CDF method. For $0 \le u \le 0.3$, $F^{-1}(u) = 0$. For $0.3 < u < 1$,*

$$u = F(x) = 1 - 0.7\mathrm{e}^{-x/5} \implies \mathrm{e}^{-x/5} = \frac{1-u}{0.7} \implies x = -5\log\left(\frac{1-u}{0.7}\right) =: F^{-1}(u).$$

*Therefore,*

| $i$ | 1 | 2 | 3 |
|---|---|---|---|
| $U_i$ | 0.4914 | 0.2845 | 0.7976 |
| $X_i = F^{-1}(U_i)$ | 1.5971 | 0.0000 | 6.2042 |

4. (15 points)
Consider the integral

$$\mu = \int_{[-1,1]^3} g(\boldsymbol{x})\mathrm{e}^{-(x_1^2+x_2^2+x_3^2)/2}\,\mathrm{d}\boldsymbol{x}.$$

Suppose that you wish to approximate $\mu$ in an unbiased way by the estimator

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{Y}_i), \qquad Y_i \overset{\text{IID}}{\sim} \boldsymbol{Y},$$

where $\boldsymbol{Y}$ is a random 3-vector. What is $f$ for

a) $\boldsymbol{Y} \sim \mathcal{U}[-1,1]^3$?

*Answer: The probability density function for this distribution is $\varrho(\boldsymbol{x}) = (1/2)^3 = 1/8$, so*

$$\mu = \int_{[-1,1]^3} \underbrace{8g(\boldsymbol{x})\mathrm{e}^{-(x_1^2+x_2^2+x_3^2)/2}}_{f(\boldsymbol{x})} \underbrace{\frac{1}{8}}_{\varrho(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}.$$

b) $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{0}, \mathsf{I})$?

*Answer: This time the probability density function for this distribution is* $\varrho(\boldsymbol{x}) = (2\pi)^{-3/2}\exp(-(x_1^2 + x_2^2 + x_3^2)/2)$ *on the domain* $\mathbb{R}^3$*, so*

$$\mu = \int_{\mathbb{R}^3} \underbrace{(2\pi)^{3/2} g(\boldsymbol{x}) \mathbb{1}_{[-1,1]^3}}_{f(\boldsymbol{x})} \underbrace{\frac{\mathrm{e}^{-(x_1^2+x_2^2+x_3^2)/2}}{(2\pi)^{3/2}}}_{\varrho(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x}.$$

c) $\boldsymbol{Y} \sim \mathcal{U}[0,1]^3$?

*Answer: This time the probability density function is* $\varrho(t) = 1$. *We first need to perform a transformation. Starting with the expression in part a), we define* $\boldsymbol{t}$ *by* $\boldsymbol{x} = 2\boldsymbol{t} - \boldsymbol{1}$. *Then,* $\mathrm{d}\boldsymbol{x} = 8\mathrm{d}\boldsymbol{t}$ *and*

$$\mu = \int_{[0,1]^3} \underbrace{8g(2\boldsymbol{t} - \boldsymbol{1})\mathrm{e}^{-[(2t_1-1)^2+(2t_2-1)^2+(2t_3-1)^2]/2}}_{f(\boldsymbol{t})} \underbrace{1}_{\varrho(\boldsymbol{t})} \, \mathrm{d}\boldsymbol{t}.$$

5. (10 points)
Let $x_0, x_1, \ldots$ denote the van der Corput sequence.

a) What is $x_{29}$?

*Answer: Since* $29 = 11101_2$*, it follows that*

$$x_{29} = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{23}{32}.$$

b) Is $\hat{\mu}_n = \frac{1}{n}\sum_{i=0}^{n-1} f(x_i)$ an unbiased estimator of $\int_0^1 f(x)\,\mathrm{d}x$ for all $n$ and all $f$? Why or why not?

*Answer: It is biased because it is not random, and e.g., for* $n = 1$, $\hat{\mu}_1 = f(0)$ *which is not the same as* $\int_0^1 f(x)\,\mathrm{d}x$ *for general* $f$.