MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This part of the final exam has FOUR questions with a maximum score of* 64 *points. Attempt them all. Added to the* 36 *point maximum on the take-home part this gives a total maximum for the final exam of* 100 *points.*

ii. *The time allowed is 120 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

vi. *Off-site students may contact the instructor at 630-696-8124.*

1. (12 points)
McDarren's Restaurant observes that B, the number of burgers in each order, follows the probability mass function

b	0	1	2	3	4
$\Pr(B = b)$	10%	40%	30%	15%	5%

A $\mathcal{U}[0, 1]$ random number generator produces the following U_i:

$$0.8638, \quad 0.2849, \quad 0.0733, \quad 0.7632, \quad 0.4527.$$

What would be the corresponding values of B_i that fit this distribution?

Answer: The CDF of B is

b	0	1	2	3	4
$\Pr(B = b)$	10%	40%	30%	15%	5%
$\Pr(B \le b)$	10%	50%	80%	95%	100%

Therefore, given any U_i, the inverse CDF gives us a B_i corresponding to the smallest b with $\Pr(B_i \le b) \ge U_i$*. So,*

i	1	2	3	4	5
U_i	0.8638	0.2849	0.0733	0.7632	0.4527
B_i	3	1	0	2	1

2. (12 points)
Consider the two-dimensional integral

$$\mu = \int_0^1 \int_0^t g(s, t) \, \mathrm{d}s \, \mathrm{d}t.$$

Given $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \overset{\text{IID}}{\sim} \mathcal{U}[0, 1]^2$, how would you approximate μ by a Monte Carlo method? (There may be more than one correct answer.)

Answer: There are a couple of ways. One is to do a variable transformation. Let $s = tu$ *and then* $\mathrm{d}s = t\mathrm{d}u$, $\boldsymbol{x} = (t, u)$, *and*

$$\mu = \int_0^1 \int_0^t g(s,t)\,\mathrm{d}s\,\mathrm{d}t = \int_0^1 \int_0^1 t g(tu,t)\,\mathrm{d}u\,\mathrm{d}t \approx \frac{1}{n}\sum_{i=1}^n X_{i1} g(X_{i1}X_{i2}, X_{i1}).$$

Another way would be using the characteristic function, $\mathbb{1}$. *and letting* $\boldsymbol{x} = (s,t)$*:*

$$\mu = \int_0^1 \int_0^t g(s,t)\,\mathrm{d}s\,\mathrm{d}t = \int_0^1 \int_0^1 \mathbb{1}_{[0,t]}(s) g(s,t)\,\mathrm{d}s\,\mathrm{d}t \approx \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{[0,X_{i2}]}(X_{i1}) g(X_{i1}, X_{i2}).$$

3. (20 points)
Consider a stock monitored monthly for three months that has a \$20 initial price, zero interest rate, and a 40% year$^{-1/2}$ volatility. Consider the IID $\mathcal{N}(0,1)$ random numbers:

$$0.0843,\ -0.1252,\ -1.2404,\ 0.7709,\ -1.8909,\ 1.1483$$

a) Use these random numbers to construct two stock paths and estimate the lookback call option price. This option expires in three months.

Answer:

b) Estimate the lookback call option price using the above random numbers and *antithetic variates.*

Answer:

4. (20 points)
Consider the following two (quasi-)Monte Carlo estimators of $\mu = \mathbb{E}[f(\boldsymbol{X})]$, where $\boldsymbol{X} \sim \mathcal{U}[0,1]^d$:

$$\hat{\mu}_{\text{IID}} = \frac{1}{n}\sum_{i=0}^{n-1} f(\boldsymbol{X}_i), \quad \boldsymbol{X}_i \overset{\text{IID}}{\sim} \mathcal{U}[0,1]^d,$$

$$\hat{\mu}_{\text{lat}} = \frac{1}{n}\sum_{i=0}^{n-1} f(\boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \mathbf{1}), \quad \{\boldsymbol{Z}_i\}_{i=0}^{n-1} \text{ is an unshifted integration lattice, } \boldsymbol{\Delta} \sim \mathcal{U}[0,1]^d.$$

a) Determine whether each of these estimators is biased or unbiased.

Answer: Both are unbiased:

$$\mathbb{E}[\hat{\mu}_{IID}] = \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}[f(\boldsymbol{X}_i)] = \frac{1}{n}\sum_{i=0}^{n-1} \mu = \mu,$$

$$\mathbb{E}[\hat{\mu}_{lat}] = \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}[f(\boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \mathbf{1})] = \frac{1}{n}\sum_{i=0}^{n-1} \mu = \mu, \text{ since } \boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \mathbf{1} \sim \mathcal{U}[0,1]^d.$$

b) What is $\mathrm{var}(\hat{\mu}_{\mathrm{IID}})$ in terms of $\mathrm{var}(f(\boldsymbol{X}))$?

Answer:

$$\mathrm{var}(\hat{\mu}_{IID}) = \frac{1}{n^2}\sum_{i=0}^{n-1}\mathrm{var}(f(\boldsymbol{X}_i)) = \frac{1}{n^2}\sum_{i=0}^{n-1}\mathrm{var}(f(\boldsymbol{X})) = \frac{\mathrm{var}(f(\boldsymbol{X}))}{n}$$

c) Is $\mathrm{var}(\hat{\mu}_{\mathrm{IID}}) = \mathrm{var}(\hat{\mu}_{\mathrm{lat}})$? Why or why not?

Answer: They are not the same

$$\begin{aligned}
\mathrm{var}(\hat{\mu}_{lat}) &= \frac{1}{n^2}\,\mathrm{var}\left(\sum_{i=0}^{n-1} f(\boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \boldsymbol{1})\right) = \frac{1}{n^2}\,\mathbb{E}\left(\sum_{i=0}^{n-1}[f(\boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \boldsymbol{1}) - \mu]\right)^2 \\
&= \frac{1}{n^2}\,\mathbb{E}\left(\sum_{i=0}^{n-1}[f(\boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \boldsymbol{1}) - \mu]\sum_{j=0}^{n-1}[f(\boldsymbol{Z}_j + \boldsymbol{\Delta} \bmod \boldsymbol{1}) - \mu]\right) \\
&= \frac{1}{n^2}\sum_{i,j=0}^{n-1}\mathbb{E}\big([f(\boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \boldsymbol{1}) - \mu][f(\boldsymbol{Z}_j + \boldsymbol{\Delta} \bmod \boldsymbol{1}) - \mu]\big) \\
&= \frac{1}{n^2}\sum_{i=0}^{n-1}\mathrm{var}\big(f(\boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \boldsymbol{1})\big) \\
&\qquad + \frac{2}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathrm{cov}\big(f(\boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \boldsymbol{1}), f(\boldsymbol{Z}_j + \boldsymbol{\Delta} \bmod \boldsymbol{1})\big) \\
&= \frac{\mathrm{var}(f(\boldsymbol{X}))}{n} + \frac{2}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathrm{cov}\big(f(\boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \boldsymbol{1}), f(\boldsymbol{Z}_j + \boldsymbol{\Delta} \bmod \boldsymbol{1})\big)
\end{aligned}$$

Since $\mathrm{cov}\big(f(\boldsymbol{Z}_i + \boldsymbol{\Delta} \bmod \boldsymbol{1}), f(\boldsymbol{Z}_j + \boldsymbol{\Delta} \bmod \boldsymbol{1})\big)$ *is not zero in general, so* $\mathrm{var}(\hat{\mu}_{IID}) \neq \mathrm{var}(\hat{\mu}_{lat})$.