MATH 565 Monte Carlo Methods in Finance

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Take-Home Part of Final Exam **Due 5 pm, Wednesday, December 10**

Instructions:

i. *This take-home part of the final exam consists of THREE questions for a total possible of* 50 marks. *Answer all of them.*

ii. *You may consult any book, web page, software repository or other inanimate object. You may use the m-files on Blackboard. You* may not *consult any other person face-to-face, by telephone, by email, or by any other means.*

iii. *Show all your work to justify your answers. Submit hard copies of your derivations, programs, output, and explanations. Answers without adequate justification will not receive credit.*

1. (15 marks)
Consider the problem of estimating $\mu = E[Y]$, where $Y = e^{X/2}$, and X is a standard normal random variable.

   a) Construct a simple Monte Carlo estimate with a confidence interval for μ using $n = 1000$ samples. Based on this simulation, how large must n be to obtain a confidence interval of half-width less than 0.001?

   b) Construct a stratified sample estimate for μ using 50 strata with 20 samples per stratum, for a total of 1000 samples. Construct a confidence interval for μ. Is this confidence interval smaller or larger than the one for simple Monte Carlo?

Answer: Note that $E[Y] = E[e^{X/2}] = e^{1/8}$. Running the program TakeHomeExamFall2008.m gives the following output:

```
Problem 1
The true expectation is 1.1331

The simple Monte Carlo estimate of the expectation
     using n = 1000 samples is 1.1175
     with a confidence interval of plus/minus 0.038848
To obtain a confidence interval of plus/minus 0.001
     requires n = 1509157 samples

The stratified sampling estimate of the expectation
     using n = 1000 samples is 1.1321
     with a confidence interval of plus/minus 0.0044992
```

The confidence interval for stratified sampling is much smaller.

2. (20 marks)
The time to get from your class on IIT's Main Campus to a job interview in downtown Chicago is $X + Y$, where X is the time in minutes to wait for a taxi and Y is the time in minutes for

the taxi ride downtown. Suppose that X has a zero-inflated exponential distribution, i.e., the cumulative probability distribution is:

$$F_X(x) = \begin{cases} 0, & -\infty < x < 0, \\ 0.4, & x = 0, \\ 1 - 0.6e^{-x/5}, & 0 < x < \infty. \end{cases}$$

Suppose that Y is a normal random variable with mean 15 minutes and variance 25 minutes squared.

a) Compute the true mean time it takes to get to your interview, $\mu = E(X + Y)$.

Answer:

$$\mu = E(X+Y) = E(X) + E(Y) = \left[0 \times 0.4 + \int_0^\infty x \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - 0.6e^{-x/5}\right) \mathrm{d}x\right] + 15$$
$$= 15 + 0.6 \int_0^\infty x \frac{1}{5} e^{-x/5} \,\mathrm{d}x = 15 + 0.6 \times 5 = 18$$

b) Perform a simple Monte Carlo simulation to estimate μ using 1000 samples. Compute $\hat{\mu}$ and a confidence interval for μ.

c) Estimate μ using 20 random scramblings of Sobol' points with 50 samples each. Compute $\hat{\mu}$ and a confidence interval for μ. *Hint: To learn how to use the Sobol' sequence generator in MATLAB's Statistics Toolbox, look at the online help or the* `samplepath.m` *program on Blackboard.*

d) Using simple Monte Carlo, estimate the probability that you will get to your interview in less than thirty minutes.

Answer: Running the program `TakeHomeExamFall2008.m` *gives the following output:*

```
Problem 2
The simple Monte Carlo estimate of the expected travel time
     using n = 1000 samples is 17.6668
     with a confidence interval of plus/minus 0.4145
The probability of arriving within 30 minutes is 0.951

The Sobol' estimate of the expected travel time
     using n = 1000 samples is 18.053
     with a confidence interval of plus/minus 0.090468
The probability of arriving within 30 minutes is 0.955
```

3. (15 marks)
Consider a Bermudan or American style put option that has a life of $T = 1$ year, and can be exercised weekly for 52 weeks. Assume the discrete time geometric Brownian motion model for the stock price with $S(0) = 100$, $r = 3\%$, $\sigma = 70\%$, and a strike price for the option of $K = 100$. Compute the option price with an error of less than one penny on the dollar using a suitable control variate. Is the price less than or greater than a Bermudan/American style put

option that can only be exercised every six months? *Hint: You may want to use the MATLAB programs on Blackboard.*

Running the program OptionPrice.m with the European option as a control variate gives the following output:

```
Using 10000 asset price samples based on a
   discrete geometric Brownian motion model of the asset
   with sampling method:
     independent and identically distributed
For an initial asset price of $100.00
   a strike price of $100.00
   1.00 years to maturity
   an interest rate of 3.00%
   a volatility of 70.00%:
For American put options monitored 52 times
Without control variates:
   the put price is $25.5621 plus/minus 0.4592
With control variates eurogbm
   the put price is $25.7855 plus/minus 0.2244
Compared to the GBM European call price of $28.4632
   and the GBM European put price of $25.5078
This computation took 0.39919 seconds

Using 10000 asset price samples based on a
   discrete geometric Brownian motion model of the asset
   with sampling method:
     independent and identically distributed
For an initial asset price of $100.00
   a strike price of $100.00
   1.00 years to maturity
   an interest rate of 3.00%
   a volatility of 70.00%:
For American put options monitored 6 times
Without control variates:
   the put price is $25.8194 plus/minus 0.4874
With control variates eurogbm
   the put price is $25.8630 plus/minus 0.1825
Compared to the GBM European call price of $28.4632
   and the GBM European put price of $25.5078
This computation took 0.13887 seconds
```

The European option is a good control variate. The option monitored less often should be less expensive, but the price difference is not much and is indistinguishable at this level of accuracy.