MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell Take-Home Final Exam Due 8 AM, Tuesday, December 3, 2013

Instructions:

i. *This take-home part of the final exam has TWO questions for a total of* 35 *points possible. You should attempt them all.*

ii. *You may consult any book, web page, software repository, notes, old tests, or other inanimate object. You may use the m-files on Blackboard. You* may not *consult any other person face-to-face, by telephone, by email, Facebook, Twitter, LinkedIn or by any other means.* **Sign here to acknowledge that you followed this instruction:**

Signature *Date*

iii. *Show all your work to justify your answers. Submit hard copies of this signed cover page and your derivations, programs, output, and explanations to me before the in-class final exam. Answers without adequate justification will not receive credit.*

iv. *In addition, as a precaution, submit soft copies of your programs to the Blackboard Dropbox. If I have difficulty understanding your computational work, I may look at your programs.*

1. (15 marks)
An extensible shifted lattice point generator takes the form of

$$\boldsymbol{x}_i = \boldsymbol{h}\phi_2(i) + \boldsymbol{\Delta} \pmod 1, \qquad i = 0, 1, \ldots,$$

where $\boldsymbol{h} \in \mathbb{N}^d$ is the generating vector, $\{\phi_2(i)\}_{i=0}^{\infty}$ is the van der Corput sequence in base 2 (the one-dimensional Sobol' sequence), and $\boldsymbol{\Delta} \in [0,1)^d$ is a shift.

a) Consider the case $\boldsymbol{\Delta} = \boldsymbol{0}$. Show that $2\boldsymbol{x}_{2^m} \bmod 1 = \boldsymbol{x}_{2^{m-1}}$ for $m = 1, 2 \ldots$.

Answer: For the van der Corput sequence in base 2, $\phi_2(2^m) = 2^{-m}$ *by definition. Thus,*

$$\begin{aligned} 2\boldsymbol{x}_{2^m} \bmod 1 &= 2(\boldsymbol{h}\phi_2(2^m) \bmod 1) \bmod 1 = 2(\boldsymbol{h}2^{-m-1} \bmod 1) \bmod 1 \\ &= 2\boldsymbol{h}2^{-m-1} \bmod 1 = \boldsymbol{h}2^{-m} \bmod 1 = \boldsymbol{h}\phi_2(2^{m-1}) \bmod 1 = \boldsymbol{x}_{2^{m-1}} \bmod 1 \end{aligned}$$

b) Dirk Nuyen's website, `http://people.cs.kuleuven.be/~dirk.nuyens/qmc-generators/` lists generating vectors for integration lattices. We choose the following 12-dimensional generator:

$$\boldsymbol{h} = (1, 182667, 469891, 498753, 110745, 446247, 250185, \ldots \\ 118627, 245333, 283199, 408519, 391023)^T$$

Consider the Asian Arithmetic Mean Put Option for a stock modeled by a geometric mean Brownian motion with initial stock price, $S(0) = 50$, strike price $K = 50$, interest rate $r = 1\%$, volatility $\sigma = 60\%$, and $T = 12$ week expiry. The stock is monitored every week and the arithmetic mean is a mean of prices at weeks $1, \ldots, 12$.

Use integration lattice sampling with $m = 30$ IID shifts $\boldsymbol{\Delta}_r \sim \mathcal{U}[0,1)^d$, of $n = 2^{10}$ samples each to price this option, i.e.,

$$\boldsymbol{x}_{i,r} = \boldsymbol{h}\phi_2(i) + \boldsymbol{\Delta}_r \pmod 1, \qquad i = 0, \ldots, 2^{10} - 1,\ r = 1, \ldots, 30.$$

Use the IID shifts and the Central Limit Theorem to estimate the error of your approximation to the option price using all 30×2^{10} samples. How large is the error at the 99% confidence level?

c) If one uses an error tolerance \$0.01 and a 99% confidence level to price this same option but with IID random sampling, how many samples are required? Which sampling scheme appears more efficient for this problem: lattice sampling or IID sampling?

d) Suppose that we change the formula for the lattices so that a different shift is applied to every sample, i.e.,

$$\boldsymbol{x}_i = \boldsymbol{h}\phi_2(i) + \boldsymbol{\Delta}_i \pmod 1, \qquad i = 0, 1, \ldots$$

with the $\boldsymbol{\Delta}_i$ IID $\mathcal{U}[0,1)^d$. What are the advantages and/or disadvantages of using this scheme for option pricing.

2. (20 marks)
Consider the problem of pricing a kind of basket call option with two stocks:

$$S_1(t) = 100\mathrm{e}^{(r-\sigma_1^2/2)t+\sigma_1 B_1(t)}, \qquad S_2(t) = 100\mathrm{e}^{(r-\sigma_2^2/2)t+\sigma_2[0.6B_1(t)+0.8B_2(t)]},$$
$$\text{discounted payoff} = \max(S_1(T) + S_2(T) - K, 0)\mathrm{e}^{-rT},$$
$$r = 1\%, \quad \sigma_1 = 40\%, \quad \sigma_2 = 60\%, \quad T = 1, \quad K = 200.$$

Price this option with IID sampling with a tolerance of \$0.05. Also price this option using any TWO of the following variance reduction techniques: control variates, importance sampling, and Sobol' sampling. Apply the two techniques you choose separately and together. Demonstrate how these techniques improve the efficiency of Monte Carlo over IID sampling in terms of number of samples and/or time required.