MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell Take-Home Final Exam Due 2 PM, Tuesday, December 9, 2014

Instructions:

i. *This take-home part of the final exam has TWO questions for a total of* 35 *points possible. You should attempt them all.*

ii. *You may consult any book, web page, software repository, notes, old tests, or other inanimate object. You may use the m-files on Blackboard. You* may not *consult any other person face-to-face, by telephone, by email, Facebook, Twitter, LinkedIn or by any other means.* **Sign here to acknowledge that you followed this instruction and return this page with your answers:**

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Signature *Date*

iii. *Show all your work to justify your answers. Submit hard copies of this signed cover page and your derivations, programs, output, and explanations to me before the in-class final exam. Answers without adequate justification will not receive credit.*

iv. *In addition, as a precaution, submit soft copies of your programs to the Blackboard Dropbox. If I have difficulty understanding your computational work, I may look at your programs.*

1. (16 points)
Consider the unshifted integration lattice node sequences, $\{\boldsymbol{z}_i\}_{i=0}^{\infty}$, described in the lecture notes. Let

$$\boldsymbol{z}_1 = \frac{(1,1,1)^T}{2}, \qquad \boldsymbol{a}_1 = (0,1,0)^T, \qquad \boldsymbol{a}_2 = (0,1,1)^T.$$

a) Find $\{\boldsymbol{z}_i\}_{i=0}^{7}$.

Answer: Here we omit the transpose sign that makes the $\boldsymbol{z}_i$ column vectors for convenience.

$$\begin{gathered}
\boldsymbol{z}_2 = \frac{\boldsymbol{z}_1 + \boldsymbol{a}_1}{2} = \frac{(1,1,1)/2 + (0,1,0)}{2} = \frac{(1,3,1)}{4}, \\
\boldsymbol{z}_4 = \frac{\boldsymbol{z}_2 + \boldsymbol{a}_2}{2} = \frac{(1,3,1)/4 + (0,1,1)}{2} = \frac{(1,7,5)}{8}, \\
\boldsymbol{z}_0 = (0,0,0), \qquad \boldsymbol{z}_3 = \boldsymbol{z}_1 + \boldsymbol{z}_2 \bmod 1 = \frac{(1,1,1)}{2} + \frac{(1,3,1)}{4} \bmod 1 = \frac{(3,1,3)}{4}, \\
\boldsymbol{z}_5 = \boldsymbol{z}_1 + \boldsymbol{z}_4 \bmod 1 = \frac{(1,1,1)}{2} + \frac{(1,7,5)}{8} \bmod 1 = \frac{(5,3,1)}{8}, \\
\boldsymbol{z}_6 = \boldsymbol{z}_2 + \boldsymbol{z}_4 \bmod 1 = \frac{(1,3,1)}{4} + \frac{(1,7,5)}{8} \bmod 1 = \frac{(3,5,7)}{8}, \\
\boldsymbol{z}_7 = \boldsymbol{z}_1 + \boldsymbol{z}_2 + \boldsymbol{z}_4 \bmod 1 = \frac{(1,1,1)}{2} + \frac{(1,3,1)}{4} + \frac{(1,7,5)}{8} \bmod 1 = \frac{(7,1,3)}{8}.
\end{gathered}$$

b) Demonstrate that the set $\{\boldsymbol{z}_i\}_{i=0}^{7}$ may be written (in a different order) as $\{i\boldsymbol{z}_4 \bmod 1\}_{i=0}^{7}$.

Answer:

i	$i\boldsymbol{z}_4 \bmod 1$	$\boldsymbol{z}_j$ for $j =?$
0	$(0,0,0)$	0
1	$\boldsymbol{z}_4$	4
2	$2\dfrac{(1,7,5)}{8} \bmod 1 = \dfrac{(1,3,1)}{4}$	2
3	$3\dfrac{(1,7,5)}{8} \bmod 1 = \dfrac{(3,5,7)}{8}$	6
4	$4\dfrac{(1,7,5)}{8} \bmod 1 = \dfrac{(1,1,1)}{2}$	1
5	$5\dfrac{(1,7,5)}{8} \bmod 1 = \dfrac{(5,3,1)}{8}$	5
6	$6\dfrac{(1,7,5)}{8} \bmod 1 = \dfrac{(3,1,3)}{4}$	3
7	$7\dfrac{(1,7,5)}{8} \bmod 1 = \dfrac{(7,1,3)}{8}$	7

c) The dual lattice for the node set $\{\boldsymbol{z}_i\}_{i=0}^7$ is defined as $P^\perp := \{\boldsymbol{k} \in \mathbb{Z}^3 : \boldsymbol{k}^T\boldsymbol{z}_i \bmod 1 = 0,\ i = 0,\dots,7\}$. Show that this is equivalent to $P^\perp := \{\boldsymbol{k} \in \mathbb{Z}^3 : \boldsymbol{k}^T\boldsymbol{z}_4 \bmod 1 = 0\}$

Answer: Note that

$$\boldsymbol{k}^T\boldsymbol{z}_i \bmod 1 = 0\ \forall i = 0,\dots,7 \iff \boldsymbol{k}^T(i\boldsymbol{z}_4) \bmod 1 = 0\ \forall i = 0,\dots,7$$
$$\iff i(\boldsymbol{k}^T\boldsymbol{z}_4) \bmod 1 = 0\ \forall i = 0,\dots,7 \iff \boldsymbol{k}^T\boldsymbol{z}_4 \bmod 1 = 0.$$

This last inequality follows by noticing that $\boldsymbol{k}^T\boldsymbol{z}_4 \bmod 1 = 0$ *implies* $i(\boldsymbol{k}^T\boldsymbol{z}_4) \bmod 1 = 0$ *for all integer* i*, and* $\boldsymbol{k}^T\boldsymbol{z}_4 \bmod 1 = 0$ *is included in the case* $i(\boldsymbol{k}^T\boldsymbol{z}_4) \bmod 1 = 0$ *for* $i = 1$.

d) Prove that

$$\frac{1}{8}\sum_{i=0}^{7} \mathrm{e}^{2\pi\sqrt{-1}\boldsymbol{k}^T\boldsymbol{z}_i} = \begin{cases} 1, & \boldsymbol{k} \in P^\perp, \\ 0, & \boldsymbol{k} \notin P^\perp. \end{cases}$$

Answer: There are a few different ways to prove this. Note that

$$\frac{1}{8}\sum_{i=0}^{7} \mathrm{e}^{2\pi\sqrt{-1}\boldsymbol{k}^T\boldsymbol{z}_i} = \frac{1}{8}\sum_{i=0}^{7} \mathrm{e}^{2\pi\sqrt{-1}i\boldsymbol{k}^T\boldsymbol{z}_4} = \frac{1}{8}\sum_{i=0}^{7} \mathrm{e}^{2\pi\sqrt{-1}i(k_1+7k_2+5k_3)/8}$$

Let $j = k_1 + 7k_2 + 5k_3 \bmod 8$*, and note that* j *is an integer. Then, we may consider this as a geometric series*

$$\frac{1}{8}\sum_{i=0}^{7} \mathrm{e}^{2\pi\sqrt{-1}\boldsymbol{k}^T\boldsymbol{z}_i} = \frac{1}{8}\sum_{i=0}^{7} \mathrm{e}^{2\pi\sqrt{-1}ij/8} = \begin{cases} 1 & j = 0 \\ \frac{1-\mathrm{e}^{2\pi\sqrt{-1}8j/8}}{8(1-\mathrm{e}^{2\pi\sqrt{-1}j/8})} = 0 & j \neq 0 \end{cases}$$

Note that $\boldsymbol{k} \in P^\perp$ *iff* $j = 0$ *by the previous part. This completes the proof.*

2. (19 points)
Consider an up and in barrier call option where the stock is modeled by a geometric Brownian motion with an initial price of \$40, an interest rate of 1%, and a volatility of 50%. The stock price path is monitored weekly for 16 weeks, which is the time to expiry of the barrier call option. The strike price is \$45 and the barrier is \$50.

   a) Use IID Monte Carlo to compute the price of this option with an error tolerance of \$0.1 with an uncertainty of 1%.
   b) Use a good importance sampling to compute the option price with the same error tolerance and uncertainty. What is a good new distribution to use? How much time does importance sampling save? How much is the number of samples reduced?
   c) Use Sobol' sampling to compute the price of this option with an error tolerance of \$0.1. Is the answer faster to compute the answer using the time stepping or Brownian bridge construction?
   d) Compute the probability to the nearest 0.002 that the discounted barrier call payout will be greater than \$5, again with a high level of confidence.