MATH 565 Monte Carlo Methods in Finance

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Due 10:30 AM, Wednesday December 9, 2015

Instructions:

i. *This test has TWO questions for a total of* 36 *points possible. You should attempt them both.*

ii. *You may consult any book, web page, software repository, notes, old tests, or other inanimate object. You may use the m-files on the Git repositories for GAIL and the class. You* may not *consult any other person face-to-face, by telephone, by email, Facebook, Twitter, LinkedIn or by any other means.* **Sign here to acknowledge that you followed this instruction and return this page with your answers:**

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Signature *Date*

iii. *Show all your work to justify your answers. Submit hard copies of this signed cover page and your derivations, programs, output, and explanations to me before the in-class final exam. Answers without adequate justification will not receive credit.*

1. (18 points)
Consider a 2-dimensional digital sequence in base 2, $\{\boldsymbol{z}_i\}_{i=0}^{\infty}$, where

$$\boldsymbol{z}_1 = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \boldsymbol{z}_2 = \left(\frac{1}{4}, \frac{3}{4}\right), \quad \boldsymbol{z}_4 = \left(\frac{1}{8}, \frac{7}{8}\right)$$

a) Compute $\{\boldsymbol{z}_i\}_{i=0}^{7}$,

Answer: Let $\oplus$ *denote digitwise addition*

i	$\boldsymbol{z}_i$
$0 = 000_2$	$0 \times \boldsymbol{z}_1 \oplus 0 \times \boldsymbol{z}_2 \oplus 0 \times \boldsymbol{z}_4 = (0,0)$
1	$\left(\frac{1}{2}, \frac{1}{2}\right) = ({}_20.100, {}_20.100)$
2	$\left(\frac{1}{4}, \frac{3}{4}\right) = ({}_20.010, {}_20.110)$
$3 = 011_2$	$\boldsymbol{z}_1 \oplus z_2 = ({}_20.110, {}_20.010) = \left(\frac{3}{4}, \frac{1}{4}\right)$
4	$\left(\frac{1}{8}, \frac{7}{8}\right) = ({}_20.001, {}_20.111)$
$5 = 101_2$	$\boldsymbol{z}_1 \oplus \boldsymbol{z}_4 = ({}_20.101, {}_20.011) = \left(\frac{5}{8}, \frac{3}{8}\right)$
$6 = 110_2$	$\boldsymbol{z}_2 \oplus \boldsymbol{z}_4 = ({}_20.011, {}_20.001) = \left(\frac{3}{8}, \frac{1}{8}\right)$
$7 = 111_2$	$\boldsymbol{z}_1 \oplus \boldsymbol{z}_2 \oplus \boldsymbol{z}_4 = ({}_20.111, {}_20.101) = \left(\frac{7}{8}, \frac{5}{8}\right)$

b) Consider the wavenumbers $\mathcal{K} = \{(0,0), (1,1), (2,2)\}$. Which wavenumbers in $\mathcal{K}$ are also in the dual net corresponding to $\{\boldsymbol{z}_i\}_{i=0}^{3}$? Which wavenumbers in $\mathcal{K}$ are also in the dual net corresponding to $\{\boldsymbol{z}_i\}_{i=0}^{7}$?

Answer: The dual net for $\{\boldsymbol{z}_i\}_{i=0}^{2^m-1}$ *is defined as*

$$\{\boldsymbol{k} \in \mathbb{N}_0^2 : \langle \boldsymbol{k}, \boldsymbol{z}_i \rangle = 0 \ \forall i = 0, \dots 2^m - 1\},$$

where

$$\langle \boldsymbol{k}, \boldsymbol{z}_i \rangle = k_{11} z_{i11} + k_{12} z_{i12} + \cdots k_{21} z_{i21} + k_{22} z_{i22} + \cdots \mod 2,$$

where $k_{j\ell}$ *are the binary digits of the* j^{th} *component of* $\boldsymbol{k}$*, and* $z_{ij\ell}$ *are the binary digits of the* j^{th} *component of* $\boldsymbol{z}_i$*. This definition can be simplified to*

$$\{\boldsymbol{k} \in \mathbb{N}_0^2 : \langle \boldsymbol{k}, \boldsymbol{z}_i \rangle = 0 \ \forall i = 1, 2, 4, \ldots, 2^{m-1}\},$$

So we check out the wavenumbers one by one:

$\boldsymbol{k}$	$\langle \boldsymbol{k}, \boldsymbol{z}_1 \rangle = \langle \boldsymbol{k}, ({}_20.1, {}_20.1) \rangle$	$\langle \boldsymbol{k}, \boldsymbol{z}_2 \rangle = \langle \boldsymbol{k}, ({}_20.01, {}_20.11) \rangle$
$(0,0)$	0	0
$(1,1) = (1_2, 1_2)$	0	1
$(2,2) = (10_2, 10_2)$	0	0

$\boldsymbol{k}$	$\langle \boldsymbol{k}, \boldsymbol{z}_4 \rangle = \langle \boldsymbol{k}, ({}_20.001, {}_20.111) \rangle$	$\in \mathcal{K}_1$	$\in \mathcal{K}_2$
$(0,0)$	0	*yes*	*yes*
$(1,1) = (1_2, 1_2)$		*no*	*no*
$(2,2) = (10_2, 10_2)$	1	*yes*	*no*

c) Consider a shift, $\boldsymbol{\Delta} = (1/3, 2/3)$. Compute $\{\boldsymbol{z}_i \oplus \boldsymbol{\Delta}\}_{i=0}^3$, where $\oplus$ denotes base 2 digit-wise addition. Your answers should be written as fractions in base 10, not just base 2 expressions.

Answer: Since $\boldsymbol{\Delta} = (1/3, 2/3) = ({}_20.010101\cdots, {}_20.101010\cdots)$*, it follows that*

i	$\boldsymbol{z}_i \oplus \boldsymbol{\Delta}$
$0 =$	$(0,0) \oplus ({}_20.0101\cdots, {}_20.1010\cdots) = ({}_20.0101\cdots, {}_20.1010\cdots) = (1/3, 2/3)$
1	$({}_20.100, {}_20.100) \oplus ({}_20.0101\cdots, {}_20.1010\cdots) = ({}_20.110101\cdots, {}_20.001010\cdots) = (5/6, 1/6)$
2	$({}_20.010, {}_20.110) \oplus ({}_20.0101\cdots, {}_20.1010\cdots) = ({}_20.000101\cdots, {}_20.011010\cdots) = (1/12, 5/12)$
$3 =$	$({}_20.110, {}_20.010) \oplus ({}_20.0101\cdots, {}_20.1010\cdots) = ({}_20.100101\cdots, {}_20.111010\cdots) = (7/12, 11/12)$

2. (18 points)
Consider an up-and-in barrier call option for a stock modeled by a geometric Brownian motion with an initial price of \$25, an interest rate of 1% year^{-1}, and a volatility of 45% year$^{-1/2}$. The stock price is monitored every two weeks. The strike price is \$25, and the time to expiry is 1/2 year. You want to design a product that has a fair price of \$2.00. What should the *barrier* be to the nearest \$0.1?

Answer: See the MATLAB script `TakeHomeAns.m`*, which can be published.*