MATH 565 Monte Carlo Methods in Finance

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Take-Home Part of Final Exam
Due 2:00 PM, Thursday, December 8, 2016

Instructions:

i. *This test has THREE questions for a total of* 36 *points possible. You should attempt them all.*

ii. *You may consult any book, web page, software repository, notes, old tests, or other inanimate object. You may use the m-files on the Git repositories for GAIL and the class. You* may not *consult any other person face-to-face, by telephone, by email, Facebook, Twitter, LinkedIn or by any other means.* **Sign here to acknowledge that you followed this instruction and return this page with your answers:**

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Signature *Date*

iii. *Show all your work to justify your answers. Submit hard copies of this signed cover page and your derivations, programs, output, and explanations to me before the in-class final exam. Answers without adequate justification will not receive credit.*

1. (6 points)
The Central Limit Theorem is often used to construct confidence intervals for means of random variables.

a) What form do those confidence intervals take?

Answer: Let $\mu = \mathbb{E}(Y)$, *let* $\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$, *and let* $\hat{\sigma}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \hat{\mu}_n)^2$, *where* $Y_1, \ldots, Y_n$ *have the same distribution as* Y. *Then*

$$\mathbb{P}\left[|\mu - \hat{\mu}_n| \le \frac{2.58 \times 1.2\hat{\sigma}_n}{\sqrt{n}}\right] \gtrsim 99\%,$$

where $1.2\hat{\sigma}_n$ *is a hopeful overestimate of* σ.

b) What conditions must be met for the Central Limit Theorem to provide reasonable confidence intervals?

Answer:

- *The* Y_i *must be IID.*
- *The variance of* Y *must be finite.*
- *The sample size must be large enough to make the Central Limit Theorem hold approximately.*
- *The fourth moment of* Y *must be finite and the sample size must be large enough so that* $1.2\hat{\sigma}_n \gtrsim \mathrm{var}(Y)$.

Be as clear and precise as you can.

2. (15 points)
The GAIL routine `gail.lattice_gen` generates the un-shifted rank-1 lattice nodesets used in the `cubLattice_g` cubature method. If you have added the GAIL repository to your path, then you should see the following output when you type the MATLAB command below:

```
>> gail.lattice_gen(1,5,3)
ans =
0         0         0
0.5000    0.5000    0.5000
0.2500    0.2500    0.2500
0.7500    0.7500    0.7500
0.1250    0.6250    0.1250
```

This output provides $\boldsymbol{z}_0, \dots, \boldsymbol{z}_4$ for a three-dimensional rank-1 lattice nodeset sequence. Note: typing `gail.lattice_gen(p,n,d)` generates $\boldsymbol{z}_{p-1}, \dots, \boldsymbol{z}_{n-1}$ of a d-dimensional nodeset sequence. The input `p` must be either

i) 1, or
ii) $2^m + 1$, where $n \le 2^{m+1} + 1$.

a) Compute $\boldsymbol{z}_6$ for this sequence and explain how it is done.

Answer: Since $6 = 110_2$*, it follows that*

$$\boldsymbol{z}_6 = 0 \times \boldsymbol{z}_1 + 1 \times \boldsymbol{z}_2 + 1 \times \boldsymbol{z}_4 \mod 1 = (0.375, 0.875, 0.375).$$

This can be verified by typing

```
>> gail.lattice_gen(1,7,3)
ans =
0         0         0
0.5000    0.5000    0.5000
0.2500    0.2500    0.2500
0.7500    0.7500    0.7500
0.1250    0.6250    0.1250
0.6250    0.1250    0.6250
0.3750    0.8750    0.3750
```

b) What is the smallest value of m for which the dual lattice corresponding to $\{\boldsymbol{z}_0, \dots, \boldsymbol{z}_{2^m-1}\}$ does *not* contain the wavenumber $\boldsymbol{k} = (1, -5, 4)$? You may use MATLAB or hand calculation to answer this question.

Answer: By hand, note that

$$(1,-5,4)\boldsymbol{z}_0 = (1,-5,4)\begin{pmatrix}0\\0\\0\end{pmatrix} \bmod 1 = 0$$

$$(1,-5,4)\boldsymbol{z}_1 = (1,-5,4)\begin{pmatrix}0.5\\0.5\\0.5\end{pmatrix} \bmod 1 = 0$$

$$(1,-5,4)\boldsymbol{z}_2 = (1,-5,4)\begin{pmatrix}0.25\\0.25\\0.25\end{pmatrix} \bmod 1 = 0$$

$$(1,-5,4)\boldsymbol{z}_4 = (1,-5,4)\begin{pmatrix}0.125\\0.625\\0.125\end{pmatrix} \bmod 1 = 0.5 \neq 0$$

See the MATLAB script `TakeHomeAns.m`*, which can be published.*

3. (15 points)
A stock is governed by a geometric Brownian motion with initial price of \$20, an interest rate of 1%, a volatility of 30%. You monitor the stock price each week for half a year (26 weeks), i.e., you compute $S(1/52), S(2/52), \ldots, S(1/2)$.

a) Use IID Monte Carlo sampling to compute the expected *range* of the stock price during this time, i.e.,

$$\mathbb{E}\left[\max_{t=0,1/52,2/52,\ldots,1/2} S(t) - \min_{t=0,1/52,2/52,\ldots,1/2} S(t)\right].$$

Compute this value within an error tolerance of 0.005.

b) Repeat your calculation, but now using Sobol' sampling. What is the difference in number of samples required and the time required in comparison to the IID Monte Carlo calculation?

Answer: See the MATLAB script `TakeHomeAns.m`*, which can be published.*