MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell **Take-Home Part of Final Exam**

Due 10:30 AM, Wednesday, December 6, 2017

Instructions:

i. *This test has FOUR questions for a total of* 36 *points possible. You should attempt them all.*

ii. *You may consult any book, web page, software repository, notes, old tests, or other inanimate object. You may use the m-files on the Git repositories for GAIL and the class. You* may not *consult any other person face-to-face, by telephone, by email, Facebook, Twitter, LinkedIn or by any other means.* **Sign here to acknowledge that you followed this instruction and return this page with your answers:**

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Signature *Date*

iii. *Show all your work to justify your answers. Submit hard copies of this signed cover page and your derivations, programs, output, and explanations to me before the in-class final exam. Answers without adequate justification will not receive credit. Calculations performed in MATLAB should be submitted as published m-files.*

1. (4 points)
Let $\hat{\mu}$ be *any* estimator for the quantity $\mu = \mathbb{E}(Y)$.

a) How does the the root mean squared error of the estimator depend on its bias and on its variance? Derive your answer.

Answer:

$$\begin{aligned}
\mathrm{RMSE}(\hat{\mu}) &= \sqrt{\mathbb{E}[(\mu - \hat{\mu})^2]} \\
&= \sqrt{\mathbb{E}[\{(\mu - \mathbb{E}(\hat{\mu})) + (\mathbb{E}(\hat{\mu}) - \hat{\mu}_n)\}^2]} \\
&= \sqrt{\mathbb{E}[(\mu - \mathbb{E}(\hat{\mu}))^2] + 2\,\mathbb{E}[(\mu - \mathbb{E}(\hat{\mu}))(\mathbb{E}(\hat{\mu}) - \hat{\mu})] + \mathbb{E}[(\mathbb{E}(\hat{\mu}) - \hat{\mu})^2]} \\
&= \sqrt{(\mu - \mathbb{E}(\hat{\mu}))^2 + 2 \times 0 + \mathrm{var}(\hat{\mu})} \\
&= \sqrt{[\mathrm{bias}(\hat{\mu})]^2 + \mathrm{var}(\hat{\mu})}
\end{aligned}$$

b) Give an example of an estimator that is unbiased. What is its variance?

Answer:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} Y_i, \quad Y_i \overset{\mathrm{IID}}{\sim} Y$$

$$\mathbb{E}(\hat{\mu}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(Y_i) = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu$$

$$\mathrm{var}(\hat{\mu}) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{var}(Y_i) = \frac{\mathrm{var}(Y)}{n}$$

c) Give an example of a zero variance estimator. What is its bias?

Answer:

$$\hat{\mu} = 0, \quad \mathbb{E}(\hat{\mu}) = 0, \quad \text{bias}(\hat{\mu}) = \mu, \quad \text{var}(\hat{\mu}) = 0$$

2. (12 points)
Suppose that the generators of a two-dimensional (unshifted) digital net are

$$\boldsymbol{z}_1 = (1/2, 1/2), \quad \boldsymbol{z}_2 = (1/4, 3/4), \quad \boldsymbol{z}_4 = (7/8, 7/8)$$

a) Compute the points $\boldsymbol{z}_0, \ldots, \boldsymbol{z}_7$, and explain how it is done.

Answer: Let $\oplus$ denote bitwise addition

i	$\boldsymbol{z}_i$
$0 = 000_2$	$0 \times \boldsymbol{z}_1 \oplus 0 \times \boldsymbol{z}_2 \oplus 0 \times \boldsymbol{z}_4 = (0, 0)$
$1 = 001_2$	$1 \times \boldsymbol{z}_1 \oplus 0 \times \boldsymbol{z}_2 \oplus 0 \times \boldsymbol{z}_4 = \boldsymbol{z}_1 = (1/2, 1/2) = ({}_20.100, {}_20.100)$
$2 = 010_2$	$0 \times \boldsymbol{z}_1 \oplus 1 \times \boldsymbol{z}_2 \oplus 0 \times \boldsymbol{z}_4 = \boldsymbol{z}_2 = (1/4, 3/4) = ({}_20.010, {}_20.110)$
$3 = 011_2$	$1 \times \boldsymbol{z}_1 \oplus 1 \times \boldsymbol{z}_2 \oplus 0 \times \boldsymbol{z}_4 = ({}_20.100, {}_20.100) \oplus ({}_20.010, {}_20.110)$
	$= ({}_20.110, {}_2.0.010) = (3/4, 1/4)$
$4 = 100_2$	$0 \times \boldsymbol{z}_1 \oplus 0 \times \boldsymbol{z}_2 \oplus 1 \times \boldsymbol{z}_4 = \boldsymbol{z}_4 = (7/8, 7/8) = ({}_20.111, {}_20.111)$
$5 = 101_2$	$1 \times \boldsymbol{z}_1 \oplus 0 \times \boldsymbol{z}_2 \oplus 1 \times \boldsymbol{z}_4 = ({}_20.100, {}_20.100) \oplus ({}_20.111, {}_20.111)$
	$= ({}_20.011, {}_2.0.011) = (3/8, 3/8)$
$6 = 110_2$	$0 \times \boldsymbol{z}_1 \oplus 1 \times \boldsymbol{z}_2 \oplus 1 \times \boldsymbol{z}_4 = ({}_20.010, {}_20.110) \oplus ({}_20.111, {}_20.111)$
	$= ({}_20.101, {}_2.0.001) = (5/8, 1/8)$
$7 = 111_2$	$1 \times \boldsymbol{z}_1 \oplus 1 \times \boldsymbol{z}_2 \oplus 1 \times \boldsymbol{z}_4 = ({}_20.100, {}_20.100) \oplus ({}_20.010, {}_20.110) \oplus ({}_20.111, {}_20.111)$
	$= ({}_20.001, {}_2.0.101) = (1/8, 5/8)$

b) The set $\{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_7\}$ is a group, which means that under digitwise addition, $\oplus$, any two points in the set added together equals one of the points in this set. Demonstrate that this is true by filling out the following 8×8 addition table:

$\oplus$	$\boldsymbol{z}_0$	$\boldsymbol{z}_1$	$\boldsymbol{z}_2$	$\boldsymbol{z}_3$	$\boldsymbol{z}_4$	$\boldsymbol{z}_5$	$\boldsymbol{z}_6$	$\boldsymbol{z}_7$
$\boldsymbol{z}_0$								
$\boldsymbol{z}_1$					$\boldsymbol{z}_5$			
$\boldsymbol{z}_2$								
$\boldsymbol{z}_3$								
$\boldsymbol{z}_4$								
$\boldsymbol{z}_5$								
$\boldsymbol{z}_6$								
$\boldsymbol{z}_7$								

For each row i and column j in the table enter the corresponding element $\boldsymbol{z}_i \oplus \boldsymbol{z}_j$. One answer has been entered for you. Fill in the other 63. Explain how you obtained your answer.

Answer: Referring to the binary digit representations in part a) and performing digitwise addition, we find that $\boldsymbol{z}_i \oplus \boldsymbol{z}_j = \boldsymbol{z}_{i \oplus j}$. *Also, note that* $i \oplus j = j \oplus i$. *So,*

$\oplus$	$0 = 000_2$	$1 = 001_2$	$2 = 010_2$	$3 = 011_2$	$4 = 100_2$	$5 = 101_2$	$6 = 110_2$	$7 = 111_2$
$0 = 000_2$	$000_2 = 0$	$001_2 = 1$	$010_2 = 2$	$011_2 = 3$	$100_2 = 4$	$101_2 = 5$	$110_2 = 6$	$111_2 = 7$
$1 = 001_2$	1	$000_2 = 0$	$011_2 = 3$	$010_2 = 2$	$101_2 = 5$	$100_2 = 4$	$111_2 = 7$	$110_2 = 6$
$2 = 010_2$	2	3	$000_2 = 0$	$001_2 = 1$	$110_2 = 6$	$111_2 = 7$	$100_2 = 4$	$101_2 = 5$
$3 = 011_2$	3	2	1	$000_2 = 0$	$111_2 = 7$	$110_2 = 6$	$101_2 = 5$	$100_2 = 4$
$4 = 100_2$	4	5	6	7	$000_2 = 0$	$001_2 = 1$	$010_2 = 2$	$011_2 = 3$
5	5	4	7	6	1	$000_2 = 0$	$011_2 = 3$	$010_2 = 2$
6	6	7	4	5	2	3	$000_1 = 0$	$001_2 = 1$
7	7	6	5	4	3	2	1	$000_2 = 0$

$\oplus$	$\boldsymbol{z}_0$	$\boldsymbol{z}_1$	$\boldsymbol{z}_2$	$\boldsymbol{z}_3$	$\boldsymbol{z}_4$	$\boldsymbol{z}_5$	$\boldsymbol{z}_6$	$\boldsymbol{z}_7$
$\boldsymbol{z}_0$	$\boldsymbol{z}_0$	$\boldsymbol{z}_1$	$\boldsymbol{z}_2$	$\boldsymbol{z}_3$	$\boldsymbol{z}_4$	$\boldsymbol{z}_5$	$\boldsymbol{z}_6$	$\boldsymbol{z}_7$
$\boldsymbol{z}_1$	$\boldsymbol{z}_1$	$\boldsymbol{z}_0$	$\boldsymbol{z}_3$	$\boldsymbol{z}_2$	$\boldsymbol{z}_5$	$\boldsymbol{z}_4$	$\boldsymbol{z}_7$	$\boldsymbol{z}_6$
$\boldsymbol{z}_2$	$\boldsymbol{z}_2$	$\boldsymbol{z}_3$	$\boldsymbol{z}_0$	$\boldsymbol{z}_1$	$\boldsymbol{z}_6$	$\boldsymbol{z}_7$	$\boldsymbol{z}_4$	$\boldsymbol{z}_5$
$\boldsymbol{z}_3$	$\boldsymbol{z}_3$	$\boldsymbol{z}_2$	$\boldsymbol{z}_1$	$\boldsymbol{z}_0$	$\boldsymbol{z}_7$	$\boldsymbol{z}_6$	$\boldsymbol{z}_5$	$\boldsymbol{z}_4$
$\boldsymbol{z}_4$	$\boldsymbol{z}_4$	$\boldsymbol{z}_5$	$\boldsymbol{z}_6$	$\boldsymbol{z}_7$	$\boldsymbol{z}_0$	$\boldsymbol{z}_1$	$\boldsymbol{z}_2$	$\boldsymbol{z}_3$
$\boldsymbol{z}_5$	$\boldsymbol{z}_5$	$\boldsymbol{z}_4$	$\boldsymbol{z}_7$	$\boldsymbol{z}_6$	$\boldsymbol{z}_1$	$\boldsymbol{z}_0$	$\boldsymbol{z}_3$	$\boldsymbol{z}_2$
$\boldsymbol{z}_6$	$\boldsymbol{z}_6$	$\boldsymbol{z}_7$	$\boldsymbol{z}_4$	$\boldsymbol{z}_5$	$\boldsymbol{z}_2$	$\boldsymbol{z}_3$	$\boldsymbol{z}_0$	$\boldsymbol{z}_1$
$\boldsymbol{z}_7$	$\boldsymbol{z}_7$	$\boldsymbol{z}_6$	$\boldsymbol{z}_5$	$\boldsymbol{z}_4$	$\boldsymbol{z}_3$	$\boldsymbol{z}_2$	$\boldsymbol{z}_1$	$\boldsymbol{z}_0$

c) Does the wavenumber $\boldsymbol{k} = (1, 2)$ belong to the dual net? Why or why not?

Answer: $(1, 2) = (01_2, 10_2)$ *does not lie in the dual net because using the notation in the notes*

$$\langle \boldsymbol{k}, \boldsymbol{z}_1 \rangle = \langle (01_2, 10_2), ({}_20.10, {}_20.10) \rangle = (1 \times 1 + 0 \times 0) + (0 \times 1 + 1 \times 0) \bmod 2 = 1 \neq 0$$

3. (8 points)
A stock is governed by a geometric Brownian motion with initial price of \$50, an interest rate of 1%, a volatility of 30%. You monitor the stock price each week for thirteen weeks (one quarter of a year) i.e., you compute $S(1/52), S(2/52), \ldots, S(1/4)$. Compute the price of an arithmetic mean call option with a strike price of \$50 with an absolute error of \$0.005.

4. (12 points)
Consider a stock under the same assumptions as in the previous problem. What is the expected number of thirteen weekly stock prices that will be over \$55 to the nearest 0.02?

Answer: See the MATLAB script `F17FinalProb3_4.m`, *which can be published.*