MATH 565 Monte Carlo Methods in Finance

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Take-Home Part of Final Exam
Due 10:30 AM, Wednesday, December 5, 2018

Instructions:

i. *This take-home part has THREE questions for a total of 36 points possible. You should attempt all questions.*

ii. *You may consult any book, web page, software repository, notes, old tests, or other inanimate object. You may use the m-files on the Git repositories for GAIL and the class. You* may not *consult any other person face-to-face, by telephone, by email, Facebook, Twitter, LinkedIn or by any other means.* **Sign here to acknowledge that you followed this instruction and return this page with your answers:**

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Signature *Date*

iii. *Show all your work to justify your answers. Submit hard copies of this signed cover page and your derivations, programs, output, and explanations to me before the in-class final exam. Answers without adequate justification will not receive credit. Neat hand-written answers are acceptable. Calculations performed in MATLAB should be submitted as published m-files.*

1. (6 points)
You want to sample random numbers X that have a probability density function PDF

$$\varrho(x) = \begin{cases} c(1-x^2), & -1 \le x \le 1, \\ 0 & x < -1 \text{ or } x > 1. \end{cases}$$

What should the value of c be? Describe how to obtain independent and identically distributed (IID) $X_1, X_2, \ldots$ from $U_1, U_2, \ldots \overset{\text{IID}}{\sim} \mathcal{U}[0,1]$. If $U_1 = 0.3456$, what is X_1?

Answer: Since

$$1 = \int_{-1}^{1} \varrho(x)\,\mathrm{d}x = \int_{-1}^{1} c(1-x^2)\,\mathrm{d}x = c\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4c}{3}$$

so $c = 3/4$. *Moreover, the CDF is*

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{4}(3x - x^3 + 2), & -1 \le x \le 1, \\ 1 & x > 1. \end{cases}$$

To find X_i *given* U_i *you must solve the equation* $F(X_i) = U_i$. *This can be done using a nonlinear equation solver. The answer is* $X_1 = -0.5260$.

2. (14 points)
Consider an *unshifted* digital sequence, $\{\boldsymbol{z}_i\}_{i=0}^{\infty}$.

a) Show that for any digital net, $\{\boldsymbol{z}_i\}_{i=0}^{2^m-1}$, the last 2^{m-1} points are a *digital shift* of the first 2^{m-1} points. What is that digital shift?

Answer: The formula for the digital net is

$$\boldsymbol{z}_i := i_0\boldsymbol{z}_1 \oplus i_1\boldsymbol{z}_2 \oplus i_2\boldsymbol{z}_4 \oplus \cdot \oplus i_{m-1}\boldsymbol{z}_{2^{m-1}}, \qquad i = i_0 + 2i_1 + 4i_2 + \cdots + 2^{m-1}i_{m-1}.$$

The last 2^{m-1} points correspond to

$$i = j + 2^{m-1} = j_0 + 2j_1 + 4j_2 + \cdots + 2^{m-2}j_{m-2} + 2^{m-1}, \qquad j = 0, \ldots, 2^{m-1} - 1,$$

and then

$$\boldsymbol{z}_{j+2^m} := j_0\boldsymbol{z}_1 \oplus j_1\boldsymbol{z}_2 \oplus j_2\boldsymbol{z}_4 \oplus \cdots \oplus j_{m-2}\boldsymbol{z}_{2^{m-2}} \oplus \boldsymbol{z}_{2^{m-1}} = \boldsymbol{z}_j \oplus \boldsymbol{z}_{2^{m-1}}.$$

Thus, the last 2^{m-1} points are a digital shift of $\boldsymbol{z}_{2^m-1}$ of the first 2^{m-1} points.

b) If $\boldsymbol{z}_{13} = (11/16, 1/16, 13/16)$ and $\boldsymbol{z}_8 = (1/16, 3/16, 11/16)$, then what is $\boldsymbol{z}_5$?

Answer: Since

$$\begin{aligned}({}_20.1011, {}_20.0001, {}_20.1101) = \boldsymbol{z}_{13} &= \boldsymbol{z}_5 \oplus \boldsymbol{z}_8 \qquad \textit{by part a)} \\ &= \boldsymbol{z}_5 \oplus ({}_20.0001, {}_20.0011, {}_20.1011),\end{aligned}$$

it follows that

$$\begin{aligned}\boldsymbol{z}_5 &= ({}_20.1011, {}_20.0001, {}_20.1101) \oplus ({}_20.0001, {}_20.0011, {}_20.1011) \\ &= ({}_20.1010, {}_20.0010, {}_20.0110) = (5/8, 1/8, 3/8)\end{aligned}$$

c) For any non-negative integers i and j, let $i \oplus j$ denote digitwise addition, e.g., $6 \oplus 5 = 110_2 + 101_2 = 011_2 = 3$. Show that $\boldsymbol{z}_{i\oplus j} = \boldsymbol{z}_i \oplus \boldsymbol{z}_j$.

Answer:

$$\begin{aligned}\boldsymbol{z}_i \oplus \boldsymbol{z}_j &= i_0\boldsymbol{z}_1 \oplus i_1\boldsymbol{z}_2 \oplus i_2\boldsymbol{z}_4 \oplus \cdots \oplus j_0\boldsymbol{z}_1 \oplus j_1\boldsymbol{z}_2 \oplus j_2\boldsymbol{z}_4 \oplus \cdots \\ &= (i_0 + j_0 \bmod 2) \oplus (i_1 + j_1 \bmod 2)\boldsymbol{z}_2 \oplus (i_2 + j_2 \bmod 2)\boldsymbol{z}_4 \oplus \\ &= k_0 \bmod 2) \oplus k_1\boldsymbol{z}_2 \oplus k_2\boldsymbol{z}_4 \oplus, \qquad \textit{where } k = i \oplus j.\end{aligned}$$

3. (16 points)
A stock is governed by a geometric Brownian motion with initial price of \$50, an interest rate of 1%, a volatility of 30%. You monitor the stock price each week for thirteen weeks (one quarter of a year) i.e., you compute $S(1/52), S(2/52), \ldots, S(1/4)$. Compute the price of a down-and-in put option with a strike price of \$50 and a barrier of \$45 with an absolute error of \$0.01 using

a) IID sampling,

b) IID sampling with a control variate: the European put option with strike price of \$50,

c) IID sampling with a different control variate: the European put option with strike price of \$45, and

d) Integration lattice sampling.

Compare the performance of these four methods and attempt to explain intuitively why certain methods perform better than others.