MATH 565 Monte Carlo Methods in Finance

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Test **Tuesday, October 16**

Instructions:

i. *This test consists of FIVE questions. Answer all of them.*

ii. *The time allowed for this test is 75 minutes*

iii. *This test is closed book, but you may use 1 double-sided letter-size sheets of notes.*

iv. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (20 points)
Let $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be i.i.d. random variables with mean μ and variance σ^2. One CPU computes the sample average $\bar{X} = (X_1 + \cdots + X_m)/m$ and another CPU computes the sample average $\bar{Y} = (Y_1 + \cdots + Y_n)/n$. You wish to combine these two sample averages to form an unbiased estimate for μ. Let $Z = a\bar{X} + b\bar{Y}$.

   a) What necessary and sufficient condition on a and b ensures that Z is *unbiased*?

   Answer: Requiring $\mu = E[Z] = aE[\bar{X}] + bE[\bar{Y}] = (a+b)\mu$ *implies that* $a + b = 1$ *is the necessary and sufficient condition.*

   b) What choice of a and b makes Z the unbiased estimate with *smallest variance*?

   Answer: Since $a + b = 1$*, it follows that*

$$\begin{aligned}
\operatorname{var}(Z) &= a^2 \operatorname{var}(\bar{X}) + b^2 \operatorname{var}(\bar{Y}) = a^2 \frac{\sigma^2}{m} + b^2 \frac{\sigma^2}{n} = \sigma^2 \left(\frac{a^2}{m} + \frac{b^2}{n} \right) \\
&= \sigma^2 \left(\frac{a^2}{m} + \frac{(1-a)^2}{n} \right) = \sigma^2 \frac{(n+m)a^2 - 2am + m}{mn} \\
&= \frac{\sigma^2 (n+m)}{mn} \left[\left(a - \frac{m}{m+n} \right)^2 + \frac{mn}{(m+n)^2} \right],
\end{aligned}$$

   Setting $a = m/(m+n)$ *minimizes the variance, so the best estimator is*

$$Z = \frac{m\bar{X} + n\bar{Y}}{m+n}.$$

   c) How does the estimator in part b) compare to the sample mean of all the random variables: $X_1, \ldots, X_m, Y_1, \ldots, Y_n$?

   Answer: It is the same.

2. (20 points)
Consider the linear congruential pseudorandom number generator

$$m_i = a m_{i-1} \mod 7, \qquad x_i = m_i/7, \qquad i = 1, 2, \ldots,$$

a) For $a = 3$, how many distinct pseudorandom numbers will be generated?

Answer: Considering the seed $m_0 = 1$, one generates

$$m_1 = 3m_0 \mod 7 = 3, \qquad m_2 = 3m_1 \mod 7 = 2, \qquad m_3 = 3m_2 \mod 7 = 6,$$
$$m_4 = 3m_3 \mod 7 = 4, \qquad m_5 = 3m_4 \mod 7 = 5, \qquad m_6 = 3m_5 \mod 7 = 1 = m_0,$$

so there are 6 distinct pseudorandom numbers.

b) Is $a = 2$ a better or worse choice than $a = 3$ for this pseudorandom number generator?

Answer: If one checks the powers of $a = 2 \mod 7$ one obtains $2^3 \mod 7 = 1 = 2^0$. Therefore, using $a = 2$ will give only three distinct pseudorandom numbers. This is a worse choice.

3. (18 points)
A uniform pseudorandom number generator produces the following output:

$$0.6557, \quad 0.0357, \quad 0.8491, \quad 0.9340, \quad 0.6787.$$

You need pseudorandom Poisson(1) numbers. A random variable $Y \sim$ Poisson(1) has the probability mass function

$$\mathbb{P}(Y = y) = \frac{e^{-1}}{y!}, \quad y = 0, 1, \ldots.$$

Use the uniform pseudorandom numbers above to *produce pseudorandom* Poisson(1) *numbers.*

Answer: We may use the inverse cumulative distribution function method. Note that

k	0	1	2	3	4	5
$\mathbb{P}(Y = k) = e^{-1}/k!$	0.3679	0.3679	0.1839	0.0613	0.0153	0.0031
$F(y) = \sum_{k=1}^{y} e^{-1}/k!$	0.3679	0.7358	0.9197	0.9810	0.9963	0.9994

The values of $F(y)$ form the endpoints of the subintervals of $[0, 1]$ which are matched to y. Therefore, the uniform pseudorandom numbers above correspond to

$$1, \quad 0, \quad 2, \quad 3, \quad 1.$$

4. (16 points)
Consider the situation of using the Monte Carlo estimator

$$Y = (\bar{X})^2, \quad \text{where } \bar{X} = (X_1 + \cdots + X_n)/n,$$

to estimate $E[X^2]$, where X, X_i are i.i.d. $\sim N(\mu, \sigma^2)$. What is the *bias* of the estimator Y, and how does it change with n? Provide an intuitive explanation.

Answer: Since $E[X^2] = \mu^2 + \sigma^2$, *and*

$$E[Y] = E[(\bar{X})^2] = \frac{1}{n^2}\sum_{i,j=1}^{n} E[X_i X_j] = \frac{\mu^2+\sigma^2}{n} + \frac{(n-1)\mu^2}{n} = \mu^2 + \frac{\sigma^2}{n},$$

therefore, $\mathrm{bias}(Y) = E[X^2 - Y] = \sigma^2(n-1)/n$, *which only vanishes for* $n = 1$. *The bias increases to* σ^2 *as* $n \to \infty$. *The reason that the bias does not vanish is that* $E[g(X)] \neq g(E[X])$.

5. (26 points)
Consider the following matrix of i.i.d. stock prices simulated every three months for one year. You will use these simulated stock prices to estimate the prices of two put options, assuming a constant interest rate of $r = 5\%$ compounded continuously, a strike price of $K = \$100$, and an expiry of $T = 1$ year.

	Time				
Path	0	3 months	6 months	9 months	12 months
1	100	89	94	95	77
2	100	73	76	79	120
3	100	85	103	85	99
4	100	92	72	63	69
5	100	76	71	63	47
6	100	113	91	103	83
7	100	92	111	115	129
8	100	99	75	72	62
9	100	101	93	111	88
10	100	112	99	106	97

a) Estimate the price of a *down and out put* option with barrier of \$70.

Answer: Paths 2 *and* 7 *are out of the money, and paths* 4, 5 *and* 8 *get knocked out, so the expected discounted payoff is approximately*

$$\frac{23+1+17+12+3}{10} \times e^{-0.05} = \$5.3.$$

b) Assuming an American put option exercise boundary of

Time	0	3 months	6 months	9 months	12 months
Boundary	72	74	77	81	100

estimate the price of the *American put* option.

Answer: At 3 *months path* 2 *is exercised. At* 6 *months paths* 4, 5, 8 *are exercised. At* 0 *and* 9 *months no paths are exercised. At* 12 *months paths* 1, 3, 6, 9, 10 *are exercised. Path* 7 *is never exercised. Thus, the estimated expected discounted payoff is*

$$\frac{1}{10}\Big(27 \times e^{-0.05/4} + (28+29+25) \times e^{-0.05/2} + (23+1+17+12+3) \times e^{-0.05}\Big) = \$16.0$$