MATH 565 Monte Carlo Methods in Finance

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Test **Wednesday, October 8**

Instructions:

i. *This test consists of FIVE questions. Answer all of them.*

ii. *The time allowed for this test is 75 minutes*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *Calculators, even of the programmable variety, are allowed, but computers are not allowed.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (20 marks)
Let X_1, X_2 be independent and identically distributed (i.i.d.) (Gaussian) random variables with mean μ and variance σ^2. Define

$$Y_1 = X_1, \quad Y_2 = \frac{X_1 + X_2}{2}.$$

Note that Y_1, Y_2 are also normal random variables.

a) What are the mean and variance of Y_2?

Answer:

$$E[Y_2] = \frac{E[X_1] + E[X_2]}{2} = \frac{\mu + \mu}{2} = \mu,$$
$$\text{var}(Y_2) = \text{var}\left(\frac{X_1 + X_2}{2}\right) = \frac{\text{var}(X_1) + \text{var}(X_2)}{4} = \frac{\sigma^2 + \sigma^2}{4} = \frac{\sigma^2}{2}.$$

b) Consider two Monte Carlo estimators for μ:

$$U = \frac{X_1 + X_2}{2}, \qquad V = \frac{Y_1 + Y_2}{2}.$$

What is the *bias* of U? What is the *bias* of V?

Answer:

$$E[U] = E[Y_2] = \mu,$$
$$E[V] = \frac{E[Y_1] + E[Y_2]}{2} = \frac{\mu + \mu}{2} = \mu.$$

Both U and V are unbiased.

c) Which estimator has a smaller *variance*: U or V?

Answer:

$$\begin{aligned}
\operatorname{var}(U) &= \operatorname{var}(Y_2) = \frac{\sigma^2}{2}, \\
\operatorname{var}(V) &= \operatorname{var}\left(\frac{Y_1+Y_2}{2}\right) = \frac{1}{4}\operatorname{var}\left(X_1 + \frac{X_1+X_2}{2}\right) \\
&= \frac{1}{4}\operatorname{var}\left(\frac{3X_1}{2} + \frac{X_2}{2}\right) = \frac{1}{4}\left(\frac{9\operatorname{var}(X_1)}{4} + \frac{\operatorname{var}(X_2)}{4}\right) \\
&= \frac{1}{4}\left(\frac{9}{4} + \frac{1}{4}\right)\sigma^2 = \frac{5}{8}\sigma^2 > \frac{\sigma^2}{2}.
\end{aligned}$$

So, U has a smaller variance.

2. (18 marks)
Let $X_1, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2. The sample average,

$$\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n),$$

is an estimate of μ and the sample standard deviation,

$$S = \sqrt{\frac{1}{n-1}\left[(X_1 - \bar{X})^2 + \cdots + (X_n - \bar{X})^2\right]},$$

tells you about the error of your estimate.

a) Suppose you observe $\bar{x} = 47.29$ and $s = 15.14$ for $n = 10000$. Construct a 95% confidence interval for μ.

Answer: The 95% *confidence interval is*

$$\bar{x} \pm \frac{1.96s}{\sqrt{n}} = 47.29 \pm \frac{1.96 \times 15.14}{\sqrt{10000}} = 47.29 \pm 0.297$$

b) If you redo the calculation again with the same n, but an independent draw of random numbers, which one or more of the following values of $\bar{x}$ would be surprising, and which one or more would be unsurprising, and why:

$$47.05, 47.29, 47.52, 50.13?$$

Answer: The answer 47.29 *is surprising because for a random algorithm you would not expect the same answer twice. The answer* 50.13 *is surprising because the* 95% *confidence interval would be about the same width of before, but then the two confidence intervals (this one and the previous) do not overlap. The other two answers are unsurprising because they even fall within the previous confidence interval.*

For the next two problems use the following string of uniform pseudorandom numbers with sample space $[0,1]$*:*

$$0.8744, 0.3540, 0.1615, 0.7563, 0.5765, 0.5678, 0.3631, 0.2645, 0.7083, 0.9752, \ldots.$$

3. (20 marks)
Machine parts have lengths of $X_1, X_2, \ldots$, which are i.i.d. independent uniform random variables with sample space $[10.30 \text{ cm}, 10.70 \text{ cm}]$. If the lengths of a pair of parts, X_{2i-1} and X_{2i}, differ by more than 0.2 cm, then the machine does not function properly.

   a) Use the pseudorandom numbers above to generate 10 successive part lengths of (5 pairs of part lengths).

   Answer: Let $Z_i \sim U[0,1]$ *be the standard uniform random numbers given above. Then* $X_i = 10.3 + 0.4U_i$*, yielding*

   $$10.6497, 10.4416, 10.3646, 10.6025, 10.5306, 10.5271, 10.4452, 10.4058, 10.5833, 10.6901.$$

   b) Use Monte Carlo simulation with 5 sample pairs to approximate the proportion of pairs of parts that do not match the desired specification.

   Answer: There are five pairs of points above. The first two are out of spec, but the last three are okay. The Monte Carlo estimate is 40%*.*

4. (18 marks)
You need pseudorandom numbers $Y_1, \ldots, Y_5$ that are i.i.d. with the the probability density function

$$f(y) = \begin{cases} 0, & -\infty < y < 0, \\ 2y, & 0 \le y < 1, \\ 0, & 1 \le y < \infty. \end{cases}$$

Use the uniform pseudorandom numbers above to produce *five pseudorandom numbers with this distribution.*

Answer: We may use the inverse cumulative distribution function method. Note that

$$F(y) = \int_{-\infty}^{y} f(t)\mathrm{d}t = \begin{cases} 0, & -\infty < y < 0, \\ y^2, & 0 \le y < 1, \\ 1, & 1 \le y < \infty. \end{cases}$$

Then, we compute $Y_i = F^{-1}(X_i) = \sqrt{X_i}$ *to get*

$$0.9351, 0.5950, 0.4018, 0.8697, 0.7592.$$

Alternatively one can use the acceptance rejection technique by accepting X_{2i-1} *if* $X_{2i-1} = f(X_{2i-1})/2 \geq X_{2i}$. *This method gives just three values of* Y_i:

$$0.8744, 0.5765, 0.3631,$$

because the second and fifth cases are rejected.

5. (24 marks)
Consider the case where an asset price, $S(t)$, with initial price \$100 is governed by a geometric Brownian motion:

$$S(t) = 100e^{-0.0950t+0.5B(t)}, \quad 0 \leq t < \infty,$$

where $B(t)$ is a standard Brownian motion ($E[B(t)] = 0$ and $\mathrm{cov}(B(t), B(\tau)) = \min(t, \tau)$). Here the risk-free interest rate is 3% and the stock volatility is 50%. A discretely monitored Asian arithmetic average call option with strike price \$100 monitored quarterly (4 times per year) and expiring in one year has a discounted payoff of

$$\text{payoff} = \max\left(\frac{1}{4}\sum_{j=1}^{4} S(j/4) - 100, 0\right) e^{-0.03}.$$

a) Given the standard *normal (Gaussian)* pseudorandom numbers

$$2.1597, 0.4500, -1.0519, 1.4813,$$

compute *one* instance of the discounted payoff using the time discretization method for the Brownian motion.

Answer: Because $B(j/4) = B((j-1)/4) + \sqrt{1/4}X_j$ *for* $j = 1, \ldots, 4$, *we have*

t	0	1/4	1/2	3/4	1
$B(t)$	0	1.0798	1.3049	0.7789	1.5196
$S(t)$	100.0	167.6	183.1	137.5	194.4

Thus, the payoff is

$$\max\left(\frac{1}{4}[167.6 + 183.1 + 137.5 + 194.4] - 100, 0\right) e^{-0.03} = 68.5.$$

b) Another nine discounted payoffs are computed by Monte Carlo simulation and their values are:

$$22.8, 0, 16.2, 58.7, 0, 0, 0, 0, 0.$$

Estimate the value of this Asian option using all ten payoffs.

Answer: The mean of these nine numbers plus the one in the previous part is \$16.6.