# MATH 565 Monte Carlo Methods in Finance

**Fred J. Hickernell** **Fall 2009**

**Test** **Monday, October 5**

*Instructions:*

i. *This test consists of FOUR questions. Answer all of them.*

ii. *The time allowed for this test is 75 minutes*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *Calculators, even of the programmable variety, are allowed. Computers, but only using MATLAB or JMP, are also allowed. No internet access.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (25 marks)
Let $X_1, \ldots, X_n, X_{n+1}, \ldots$ be independent and identically distributed random variables with mean $\mu$ and variance $\sigma^2$. Define the sample mean of the first $n$ random variables by

$$\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n).$$

a) Derive the mean and variance of $\bar{X}$. Is $\bar{X}$ a biased or unbiased estimator of $\mu$?

*Answer:*

$$E[\bar{X}] = \frac{1}{n}E\{X_1 + \cdots + X_n\} = \frac{n\mu}{n} = \mu,$$
$$\mathrm{var}(\bar{X}) = \frac{1}{n^2}\mathrm{var}\left(X_1 + \cdots + X_n\right) = \frac{1}{n^2}\left[\mathrm{var}(X_1) + \cdots + \mathrm{var}(X_n)\right] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

*Thus, $\bar{X}$ is an unbiased estimator of $\mu$.*

b) Define the sample mean of the next $m$ random variables $X_i$ by

$$Y = \frac{1}{m}(X_{n+1} + \cdots + X_{n+m}).$$

We may think of $\bar{X}$ and $Y$ as two Monte Carlo estimators of $\mu$. Which of these has the smaller variance?

*Answer: Since* $\mathrm{var}(Y) = \sigma^2/m$, *then $\bar{X}$ hast the smaller variance iff $n > m$, and $Y$ has the smaller variance iff $m > n$.*

c) Let $Z = a\bar{X} + (1-a)Y$ for some choice of constant $a$. This is a third Monte Carlo estimator of $\mu$ derived from first two estimators. Show that $Z$ is unbiased, regardless of the choice of $a$.

*Answer:*

$$E[Z] = E[a\bar{X} + (1-a)Y] = aE[\bar{X}] + (1-a)E[Y] = a\mu + (1-a)\mu = \mu.$$

d) For what choice of $a$ does the unbiased Monte Carlo estimator $Z$ have minimum variance, and what is this minimum variance?

*Answer:*

$$\begin{aligned}
\operatorname{var}(Z) &= \operatorname{var}\left(a\bar{X} + (1-a)Y\right) = a^2 \operatorname{var}(\bar{X}) + (1-a)^2 \operatorname{var}(Y) \\
&= a^2 \frac{\sigma^2}{n} + (1-a)^2 \frac{\sigma^2}{m} = \sigma^2 \left[\frac{a^2}{n} + \frac{1 - 2a + a^2}{m}\right] \\
&= \sigma^2 \left[a^2 \left(\frac{1}{n} + \frac{1}{m}\right) - \frac{2a}{m} + \frac{1}{m}\right] = \frac{\sigma^2}{m} \left[a^2 \left(\frac{n+m}{n}\right) - 2a + 1\right] \\
&= \sigma^2 \left(\frac{n+m}{nm}\right) \left[\left(a - \frac{n}{n+m}\right)^2 - \left(\frac{n}{n+m}\right)^2 + \frac{n}{n+m}\right] \\
&= \sigma^2 \left(\frac{n+m}{nm}\right) \left[\left(a - \frac{n}{n+m}\right)^2 + \frac{mn}{(n+m)^2}\right] \\
&= \sigma^2 \left[\left(\frac{n+m}{nm}\right) \left(a - \frac{n}{n+m}\right)^2 + \frac{1}{n+m}\right]
\end{aligned}$$

*The minimum variance occurs for* $a = n/(n+m)$ *and* $1 - a = m/(n+m)$, *so*

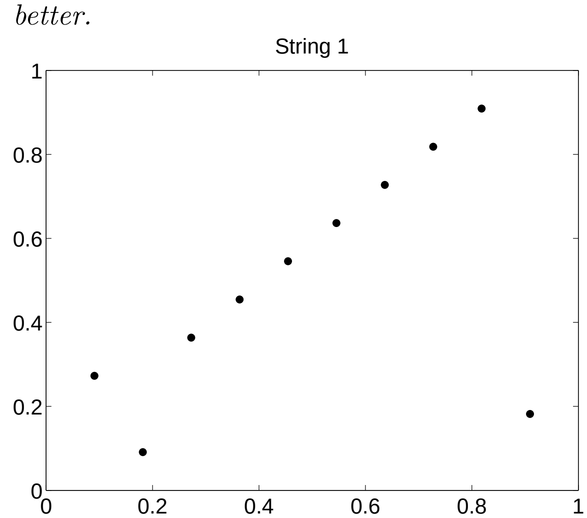
$$Z = a\bar{X} + (1-a)Y = \frac{1}{n+m}\left(X_1 + \cdots + X_n + X_{n+1} + \cdots + X_{n+m}\right),$$

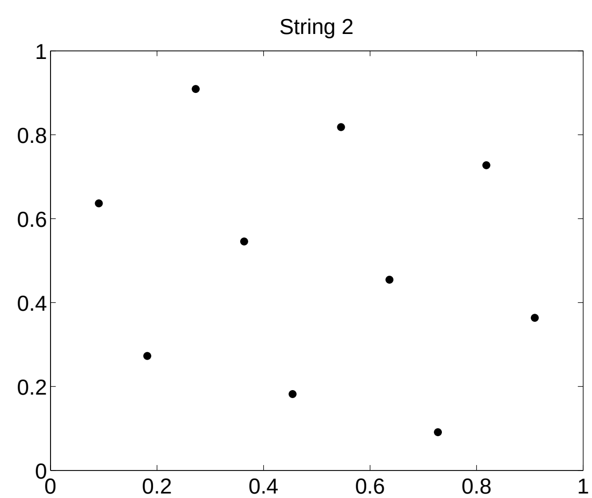
*and* $\operatorname{var}(Z) = \sigma^2/(n+m)$.

2. (20 marks)
Consider the two periodic strings of uniform pseudorandom numbers. Although neither is perfect, *which one is better and why?*

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| String 1 | 0.364 | 0.455 | 0.545 | 0.636 | 0.727 | 0.818 | 0.909 | 0.182 | 0.091 | 0.273 | 0.364 | ⋯ |
| String 2 | 0.364 | 0.545 | 0.818 | 0.727 | 0.091 | 0.636 | 0.455 | 0.182 | 0.273 | 0.909 | 0.364 | ⋯ |

*Answer: If we plot the ordered pairs* $(x_i, x_{i+1})$, $i = 1, 2, \ldots$, *the second string fills the square better than the first, in which most of these points lie on a single line. Thus, the second string is better.*



3. (25 marks)
Consider the situation where the probability of having at least one taxi waiting at the taxi stand is 50%, in which case the waiting time is 0. If there is no taxi waiting at the taxi stand, then the waiting time for the next taxi to arrive is exponential with a mean of 2 minutes. This means that the cumulative distribution function for the waiting time for a taxi, $T$, is

$$F(t) = \text{Prob}(T \le t) = \begin{cases} 0, & -\infty < t < 0. \\ 1 - \frac{1}{2}e^{-t/2}, & 0 \le t < \infty. \end{cases}$$

Use the String 1 of pseudorandom numbers in the previous problem to produce *five pseudo-random numbers with this distribution.*

*Answer: We may use the inverse cumulative distribution function method. Let $x = F(t)$, and $t = F^{-1}(x)$. Then*

$$F^{-1}(x) = \begin{cases} 0, & 0 \le< x \le 0.5, \\ -2\log(2(1-x)), & 0.5 < x < 1. \end{cases}$$

*Then, we compute $T_i = F^{-1}(X_i)$ to get*

| *String 1* $X_i$ | 0.364 | 0.455 | 0.545 | 0.636 | 0.727 | $\cdots$ |
|---|---|---|---|---|---|---|
| $T_i$ | 0 | 0 | 0.189 | 0.635 | 1.210 | $\cdots$ |

4. (30 marks)
There are two stocks whose prices right now are both \$100, and whose prices one year later will be $S_j(1) = 100e^{(0.03-\sigma_j^2/2)+\sigma_j X_j}$ for $j = 1, 2$, where the $X_1$ and $X_2$ are i.i.d. $N(0,1)$ random variables, and $\sigma_1 = 0.5$ and $\sigma_2 = 0.9$. A basket call option pays

$$\max(\max(S_1(1), S_2(1)) - 100, 0)e^{-0.03}$$

in today's dollars. Perform a Monte Carlo simulation to compute the fair price of this option with a relative error of no more than 1%, i.e., the 95% confidence interval should have a half width satisfying this tolerance. How many samples do you need? Write down your program or the pseudo-code for your algorithm as well as the results of your simulation.

*Answer: The code for this program plus the output is given below. The value of $n$ may be found by choosing a trial value, $n_0$, obtaining a relative error estimate $\varepsilon_0$, and then estimating $n = 1.2n_0(\varepsilon_0/0.01)^2$. The value $n = 100000$ is adequate.*

```
% Basket option
r=0.03; %interest rate
sig1=0.5; sig2=0.9; %volatilities
n=1e3 %number of samples
tol=0.01;
%compute stock prices & payoff
stock1=100*exp((r-sig1*sig1/2)+sig1*randn(n,1));
stock2=100*exp((r-sig2*sig2/2)+sig2*randn(n,1));
payoff=max(max(stock1,stock2)-100,0)*exp(-r);
```

```
price=mean(payoff) %MC estimate of fair price
ciwidth=1.96*std(payoff)/sqrt(n) %confidence interval half-width
relerr=ciwidth/price %estimate relative error
nest=1.2*n*(relerr/tol)^2 %estimated n to meet tolerance

n =
        1000
price =
   53.3233
ciwidth =
    6.1874
relerr =
    0.1160
nest =
   1.6157e+05
n =
      200000
price =
   52.5749
ciwidth =
    0.4354
relerr =
    0.0083
nest =
   1.6458e+05
```