MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell **Fall 2009**

Make-Up Test **Friday, October 9**

Instructions:

i. *This test consists of FOUR questions. Answer all of them.*

ii. *The time allowed for this test is 75 minutes*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *Calculators, even of the programmable variety, are allowed. Computers, but only using MATLAB or JMP, are also allowed. No internet access.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (25 marks)
Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2. Define the *weighted* sample mean by

$$\bar{X} = w_1 X_1 + \cdots + w_n X_n,$$

for deterministic real-valued weights, $w_1, \ldots, w_n$.

a) Derive the mean and variance of $\bar{X}$.

Answer:

$$E(\bar{X}) = E(w_1 X_1 + \cdots + w_n X_n) = w_1 \mu + \cdots + w_n \mu = \mu(w_1 + \cdots + w_n),$$
$$\operatorname{var}(\bar{X}) = \operatorname{var}(w_1 X_1 + \cdots + w_n X_n) = w_1^2 \operatorname{var}(X_1) + \cdots + w_n^2 \operatorname{var}(X_n) = \sigma^2 (w_1^2 + \cdots + w_n^2).$$

b) Under what condition on the weights is $\bar{X}$ an unbiased estimator of μ?

Answer: The estimator $\bar{X}$ is unbiased if $w_1 + \cdots + w_n = 1$.

c) What choice of weights makes $\operatorname{var}(\bar{X})$ minimum? Is $\bar{X}$ unbiased for this choice of weights?

Answer: If $w_1 = \cdots = w_n = 0$, then $\operatorname{var}(\bar{X}) = 0$, the smallest value possible. In this case $\bar{X} = 0$, and so $\bar{X}$ is biased (unless $\mu = 0$).

d) What choice of the weights gives an unbiased estimator with minimum variance?

Answer:

$$\begin{aligned}
\operatorname{var}(\bar{X}) &= \sigma^2 (w_1^2 + \cdots + w_n^2) \\
&= \sigma^2 \left\{ \left[(w_1 - 1/n)^2 + \cdots + (w_n - 1/n)^2 \right] + 2(w_1 + \cdots + w_n)/n - (1 + \cdots + 1)/n^2 \right\} \\
&= \sigma^2 \left\{ \left[(w_1 - 1/n)^2 + \cdots + (w_n - 1/n)^2 \right] + 2/n - 1/n \right\} \\
&= \sigma^2 \left\{ \left[(w_1 - 1/n)^2 + \cdots + (w_n - 1/n)^2 \right] + 1/n \right\} \geq \sigma^2/n
\end{aligned}$$

Note that this lower bound can be reached if $w_1 = \cdots = w_n = 1/n$, i.e., equal weights give minimum variance.

2. (20 marks)
Consider linear congruential generator

$$x_0 = m_0/7, \quad x_i = ax_{i-1} \mod 1, \; i = 1, 2, \ldots,$$

where the seed, m_0, is some integer between 1 and 6, and a is also some integer between 1 and 6. For what values of a will this random number generator have the largest period possible. What is this maximum period?

Answer: The maximum period possible is $6 = 7 - 1$. *We can check the possible values of* a *with any seed (say* $m_0 = 1$*) to see what the period is:*

a	x_0	x_1	x_2	x_3	x_4	x_5	x_6	$\cdots$
1	1/7	1/7	1/7	1/7	1/7	1/7	1/7	$\cdots$
2	1/7	2/7	4/7	1/7	2/7	4/7	1/7	$\cdots$
3	1/7	3/7	2/7	6/7	4/7	5/7	1/7	$\cdots$
4	1/7	4/7	2/7	1/7	4/7	2/7	1/7	$\cdots$
5	1/7	5/7	4/7	6/7	2/7	3/7	1/7	$\cdots$
6	1/7	6/7	1/7	6/7	1/7	6/7	1/7	$\cdots$

There will be a full period for $a = 3, 5$.

3. (25 marks)
Consider a random variable X with the geometric distribution with mean 2:

$$\text{Prob}(X = x) = 2^{-x}, \quad x = 1, 2, \ldots.$$

You may think of X as denoting the number of tries that it takes for an email message to be passed successfully to the recipient if the chance of a success on each try is 1/2, independent of every other try. Use the linear congruential generator in the previous problem with $a = 3$ and $m_0 = 1$ to produce *five pseudorandom numbers with this distribution.*

Answer: First we find the cumulative distribution function:

$$F(x) = \text{Prob}(X \le x) = \sum_{y=1}^{x} \text{Prob}(X = y) = \sum_{y=1}^{x} 2^{-y} = 1 - 2^{-x}, \quad x = 1, 2, \ldots.$$

Then we use the inverse cumulative distribution function method. Let $x = F(t)$, *and* $t = F^{-1}(x)$. *Then*

$$F^{-1}(x) = \begin{cases} 1, & 0 \le x \le 1/2, \\ 2, & 0.5 < x \le 0.75, \\ \cdots & \\ t, & 1 - 2^{-t+1} < x \le 1 - 2^{-t}, \\ \cdots. & \end{cases}$$

Then, we compute $T_i = F^{-1}(X_i)$ *to get*

X_i	1/7	3/7	2/7	6/7	4/7	5/7$\cdots$	
T_i	1	1	1	3	2	2	$\cdots$

4. (30 marks)
Let X_1, X_2 are i.i.d. uniform random variables on the interval $[0, 1]$. Use Monte Carlo simulation to estimate hat the probability to the nearest 1% that $\min(X_1, X_2) \leq 0.5 \leq \max(X_1, X_2)$.

Answer: The code for this program plus the output is given below. The value of n may be found by choosing a trial value, n_0, obtaining an absolute error estimate ε_0, and then estimating $n = 1.2n_0(\varepsilon_0/0.01)^2$. The value $n = 12000$ is adequate. The true value is 50%.

```
% Probability of bracketing
n=1e3 %number of samples
tol=0.01; %error tolerance
%compute ordered pairs of random variables
x=rand(n,2); %generate ordered pairs of uniform numbers
bracket=(min(x,[],2)<=0.5)&(max(x,[],2)>=0.5);
prob=mean(bracket) %MC estimate of probability of bracketing
ciwidth=1.96*sqrt(prob*(1-prob))/sqrt(n) %confidence interval half-width
nest=1.2*n*(ciwidth/tol)^2 %estimated n to meet tolerance
```

```
n =
        1000
prob =
    0.4990
ciwidth =
    0.0310
nest =
   1.1525e+04
n =
       12000
prob =
    0.5061
ciwidth =
    0.0089
nest =
   1.1523e+04
```