MATH 565 Monte Carlo Methods in Finance

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Test **Tuesday, October 5**

Instructions:

i. *This test consists of FOUR questions. Answer all of them.*

ii. *The time allowed for this test is 75 minutes*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *Calculators, even of the programmable variety, are allowed. Computers, but only using MATLAB or JMP, are also allowed. No internet access.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (27 marks)
Let $X_1, \ldots, X_n, X_{n+1}, \ldots$ be independent and identically distributed random variables, and let $Y = g(X)$ denote the payoff of an option. The function g takes on values in the interval $[0, \infty)$. You are interested to know $\rho = \mathbb{P}[g(X) > 0]$, i.e., the proportion of the payoffs that are positive (not zero). Let Z_n be a Monte Carlo estimator for ρ, i.e.,

$$Z_n = \frac{1}{n}\sum_{i=1}^{n} 1_{(0,\infty)}(g(X_i)), \quad \text{where } 1_{(0,\infty)}(y) = \begin{cases} 0, & -\infty < y \le 0, \\ 1, & 0 < y < \infty, \end{cases}.$$

a) Show that Z_n is an unbiased estimator of ρ.

Answer:

$$\begin{aligned} E[Z_n] &= E\left[\frac{1}{n}\sum_{i=1}^{n} 1_{(0,\infty)}(g(X_i))\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} E\left[1_{(0,\infty)}(g(X_i))\right] \qquad (\textit{take } E \textit{ inside the sum}) \\ &= \frac{1}{n}\sum_{i=1}^{n} \rho \qquad \left(E\left[1_{(0,\infty)}(g(X_i))\right] = 1 \times \mathbb{P}[g(X_i) > 0] = \rho\right) \\ &= \rho \end{aligned}$$

Thus, Z_n is an unbiased estimator of ρ.

b) Show that the variance of Z_n is $\rho(1-\rho)/n$.

Answer: The simplest way is to compute

$$\begin{aligned} \operatorname{var}[Z_n] &= \operatorname{var}\left[\frac{1}{n}\sum_{i=1}^{n} 1_{(0,\infty)}(g(X_i))\right] \\ &= \frac{1}{n^2}\sum_{i=1}^{n} \operatorname{var}[1_{(0,\infty)}(g(X_i))] \quad (\textit{since the } 1_{(0,\infty)}(g(X_i)) \textit{ are mutually independent}) \end{aligned}$$

Now,

$$\begin{aligned}\text{var}[1_{(0,\infty)}(g(X_i))] &= E[\{1_{(0,\infty)}(g(X_i))\}^2] - \{E[1_{(0,\infty)}(g(X_i))]\}^2 \\ &= \rho - \rho^2 = \rho(1-\rho).\end{aligned}$$

Thus,

$$\text{var}[Z_n] = \frac{1}{n^2} n\rho(1-\rho) = \frac{\rho(1-\rho)}{n}$$

Another way to derive this is:

$$\begin{aligned}E[Z_n^2] &= E\left[\left\{\frac{1}{n}\sum_{i=1}^{n} 1_{(0,\infty)}(g(X_i))\right\}^2\right] \\ &= E\left[\left\{\frac{1}{n}\sum_{i=1}^{n} 1_{(0,\infty)}(g(X_i))\right\}\left\{\frac{1}{n}\sum_{j=1}^{n} 1_{(0,\infty)}(g(X_j))\right\}\right] \\ &= E\left[\frac{1}{n^2}\sum_{i,j=1}^{n} 1_{(0,\infty)}(g(X_i))1_{(0,\infty)}(g(X_j))\right] \\ &= \frac{1}{n^2}\sum_{i,j=1}^{n} E\left[1_{(0,\infty)}(g(X_i))1_{(0,\infty)}(g(X_j))\right] \qquad \textit{(take E inside the sum).}\end{aligned}$$

Note that

$$E\left[1_{(0,\infty)}(g(X_i))1_{(0,\infty)}(g(X_j))\right] = \begin{cases} E\left[1_{(0,\infty)}(g(X_i))\right] = \rho, & i = j \\ E\left[1_{(0,\infty)}(g(X_i))\right]E[1_{(0,\infty)}(g(X_j))\right] = \rho^2, & i \neq j.\end{cases}$$

Thus,

$$\begin{aligned}E[Z_n^2] &= \frac{1}{n^2}\left[\sum_{\substack{i,j=1 \\ i=j}}^{n} \rho + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \rho^2\right] \\ &= \frac{n\rho + (n^2-n)\rho^2}{n^2} = \frac{\rho[1-(n-1)\rho]}{n} \\ \text{var}(Z_n) = E[Z_n^2] - [E(Z_n)]^2 &= \frac{\rho[1-(n-1)\rho]}{n} - \rho^2 = \frac{\rho(1-\rho)}{n}.\end{aligned}$$

c) To construct a 95% confidence interval for ρ in terms of of Z_n that has a half-width of $0.01 = 1\%$, how large must n be, independent of ρ?

Answer: The half-width of the confidence interval is

$$1.96\sqrt{\text{var}(Z_n)} = 1.96\sqrt{\frac{\rho(1-\rho)}{n}} \leq \frac{1.96}{2\sqrt{n}} \qquad \textit{(attained when } \rho = 1/2\textit{)}$$

Thus, we need

$$\frac{1.96}{2\sqrt{n}} \leq 0.01 \iff n \geq \left(\frac{1.96}{2 \times 0.01}\right)^2 = 9604 \approx 10000$$

2. (25 marks)
Consider the situation in the previous problem where the payoff is for a European call option with an initial price of \$100, a strike price of \$120, a risk-free interest rate of 1%, a volatility of 50%, and an expiry date of 1 year,

$$g(X) = \max(100e^{-0.115+0.5X} - 120, 0)e^{-0.01}, \quad X \sim N(0,1).$$

Estimate ρ, the proportion of positive payoffs, with an approximate 95% confidence interval of half-width 1%.

Answer: One could solve this analytically:

$$\begin{aligned}
\mathbb{P}[100e^{-0.115+0.5X} - 120 > 0] &= \mathbb{P}\left[e^{-0.115+0.5X} > \frac{120}{100} = 1.2\right] \\
&= \mathbb{P}[-0.115 + 0.5X > \log(1.2)] \\
&= \mathbb{P}[X > 2\{\log(1.2) + 0.115\}] \\
&= 1 - \mathbb{P}[X \le 2\{\log(1.2) + 0.115\}] \\
&= 1 - \Phi(2\{\log(1.2) + 0.115\}) \approx 27.60\%
\end{aligned}$$

However, I expected you to solve it by Monte Carlo. The MATLAB program that solves this problem is

```
%% Problem 2
S0=100; %initial price
K=120; %strike price
r=0.01; %interest rate
sig=0.5; %volatility
T=1; %time to expiry
n=10000; %sample size computed from problem 1
x=randn(n,1); %normal random variables
pospayoff=S0*exp((r-sig^2/2)*T + sig*sqrt(T)*x)>K; %positive payoff
rhohat=mean(pospayoff); %estimated proportion of positive payoffs
varrhohat=rhohat*(1-rhohat)/n; %estimated variance of estimator
ciwidth=1.96*sqrt(varrhohat); %half-width of the confidence interval
disp('Problem 2')
disp(['The proportion of positive payoffs is '...
    num2str(100*rhohat) '% +/- '...
    num2str(100*ciwidth) '%'])

Problem 2
The proportion of positive payoffs is 27.36% +/- 0.87378%
```

3. (27 marks)
The Cauchy distribution has the probability density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

a) Explain how to get i.i.d. Cauchy random numbers $X_1, X_2, \ldots$ from i.i.d. uniform random numbers $U_1, U_2, \ldots$, which are uniform on $[0, 1]$.

Answer: The cumulative distribution function is

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)} \, \mathrm{d}t \\
&= \frac{\tan^{-1}(t)}{\pi}\bigg|_{-\infty}^{x} \\
&= \frac{\tan^{-1}(x) - \tan^{-1}(-\infty)}{\pi} = \frac{\tan^{-1}(x) + \pi/2}{\pi} \\
&= \frac{1}{2} + \frac{\tan^{-1}(x)}{\pi}.
\end{aligned}$$

It's inverse is derived as

$$u = \frac{1}{2} + \frac{\tan^{-1}(x)}{\pi} \iff x = \tan(\pi[u - 1/2]) = F^{-1}(u)$$

Thus, $X_i = \tan(\pi[U_i - 1/2])$ *gives Cauchy random numbers.*

b) Using part a), estimate

$$\int_{-\infty}^{\infty} \frac{|x|^{1/4}}{\pi(1+x^2)} \, \mathrm{d}x$$

by Monte Carlo with a sample size of $n = 10^5$. Give a 95% confidence interval for your estimate.

Answer: The MATLAB program for this problem is

```
%% Problem 3
n=100000; %sample size
u=rand(n,1); %uniform random numbers
x=tan(pi*(u-1/2)); %Cauchy random numbers
y=abs(x).^(1/4);
meany=mean(y);
stdy=std(y);
ciwidth=1.96*stdy/sqrt(n);
disp('Problem 3')
disp(['The integral is ' ...
    num2str(meany) ' +/- ' num2str(ciwidth)])
```

```
Problem 3
The integral is 1.0824 +/- 0.003039
```

4. (21 marks)
Consider an American put option with a strike price of \$100, and the case of zero interest. At some time t before expiry, the value of continuing to *hold* the option, H, given an asset price of $S(t)$ is estimated by regression to be $H(S(t))$, where

$$H(x) = 0.8 \max(105 - x, 0).$$

a) Determine the exercise boundary at time t, i.e., the value of b for which the option should be exercised if $S(t) < b$.

Answer: The value of exercising the option if the asset price is $S(t)$ is given by $P(S(t))$, where the payoff function, P is given by $P(x) = \max(100 - x, 0)$. Thus,

$$P(x) > H(x) \iff \max(100 - x, 0) > 0.8 \max(105 - x, 0) = \max(84 - 0.8x, 0)$$
$$\iff 100 - x > 84 - 0.8x \iff 16 > 0.2x \iff x < 80.$$

Therefore, the boundary is $b = \$80$.

b) A Monte Carlo simulation generates following scenarios,

$S(t)$	88	98	121	75	113	104	67	95	...

Determine the expected values of the put option given the above values of $S(t)$, assuming that the option has not been exercised prior to time t.

Answer: The expected value is the maximum of the value of exercising an holding, which is the last row in the table below.

$S(t)$	88	98	121	75	113	104	67	95	...
$P(S(t))$	12	2	0	25	0	0	33	5	...
$H(S(t))$	13.6	5.6	0	24	0	0.8	30.4	8	...
$\max(P(S(t)), H(S(t)))$	13.6	5.6	0	25	0	0.8	33	8	...