MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell **Fall 2011**

Test **Tuesday, October 11**

Instructions:

i. *This test consists of FOUR questions. Answer all of them.*
ii. *The time allowed for this test is 75 minutes*
iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*
iv. *Calculators, even of the programmable variety, are allowed. Computers, but only using MATLAB or JMP, are also allowed. No internet access.*
v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (30 marks)
Consider the problem of numerically evaluating the integral

$$\mu = \int_0^3 \int_0^3 f(x_1, x_2)\,\mathrm{d}x_1, \mathrm{d}x_2,$$

where $f : [0,3]^2 \to \mathbb{R}$ is some integrand. A Monte Carlo estimate of this integral is given by

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} cf(\boldsymbol{X}_i)$$

where $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are i.i.d. random vectors, and c is a constant to be determined below.

a) Your random number generator gives you $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ i.i.d. uniform random vectors on $[0,1]^2$. How should you construct the $\boldsymbol{X}_i$ from the $\boldsymbol{U}_i$?

Answer: Since we want $\boldsymbol{X}_i$ to lie on $[0,3]^2$, we choose $\boldsymbol{X}_i = 3\boldsymbol{U}_i$.

b) What choice of c makes $\hat{\mu}$ unbiased?

Answer:

$$\begin{aligned}
E[\hat{\mu}] &= \frac{1}{n}\sum_{i=1}^{n} cE[f(\boldsymbol{X}_i)] = \frac{1}{n}\sum_{i=1}^{n} cE[f(3\boldsymbol{U}_i)] = cE[f(3\boldsymbol{U}_1)] \qquad \textit{(since the } U_i \textit{ are i.i.d.)} \\
&= c\int_0^1\int_0^1 f(3u_1, 3u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 = c\int_0^3\int_0^3 f(x_1,x_2)\,\frac{\mathrm{d}x_1}{3}\,\frac{\mathrm{d}x_2}{3} \qquad (x_1 = 3u_1,\ x_2 = 3u_2) \\
&= \frac{c}{9}\int_0^3\int_0^3 f(x_1,x_2)\,\mathrm{d}x_1, \mathrm{d}x_2 = \frac{c\mu}{9}
\end{aligned}$$

Thus, one should choose $c = 9$ to obtain an unbiased estimate.

2. (30 marks)
   Consider the previous problem with the integrand chosen as

$$f(x_1, x_2) = e^{-x_1^2+x_1x_2-x_2^2}.$$

   Use a Monte Carlo method to estimate the integral in question with an absolute error of no more than 0.01. Use an appropriate error estimate to insure that the sample size you choose is large enough. How large a sample size do you need to use? *Hint: You may want to test your program for the function $f(x_1, x_2) = 1$. What should the error be for this integrand?*

   Answer: One way to rewrite the integral is as

$$\mu = \int_0^3 \int_0^3 9f(x_1, x_2) \frac{\mathrm{d}x_1}{3} \frac{\mathrm{d}x_2}{3}$$

   In this way we see that we are integrating the function $9f$ with respect to the uniform distribution over $[0, 3]^2$. The answer from the program below is about $\hat{\mu} = 1.206$ requiring about 210 000 samples. Moreover, $\hat{\sigma} \approx 1.968$ (Your answer may vary.)

```
f=@(x) 9*exp(-x(:,1).*x(:,1)+x(:,1).*x(:,2)-x(:,2).*x(:,2)); %integrand
tol=1e-2; %error tolerance
n0=1000; %initial sample size
x=3*rand(n0,2); %initial sample
fx=f(x); %integrand values
muhat0=mean(fx); %initial sample mean (not necessary)
sig0=std(fx); %estimate of standard deviation
n=ceil(1.5*(1.96*sig0/tol)^2) %final sample size
x=3*rand(n,2); %final sample
fx=f(x); %integrand values
muhat=mean(fx) %estimate of integral
```

3. (20 marks)
   The Brownian motion, $B(t)$, defined for $t \geq 0$ is a very important stochastic process (random function). For every fixed t, $B(t)$ is a Gaussian (normal) random variable, and in addition,

$$E[B(t)] = 0, \quad \operatorname{cov}(B(t), B(s)) = \min(t, s), \qquad \forall t, s \geq 0,$$

   Use these properties of the Brownian motion to evaluate the quantities below:

   i) $E\{[B(t)]^2\}$
   ii) $B(0)$
   iii) $\operatorname{cov}(B(1), B(2) - B(1))$
   iv) $E[B(2)|B(1) = 0.5]$ (the expected value of $B(2)$ given that $B(1) = 0.5$)

   Answer:

$$E\{[B(t)]^2\} = \operatorname{var}(B(t)) + \{E[B(t)]\}^2 = \operatorname{cov}(B(t), B(t)) + 0 = \min(t, t) = t$$

Since $E\{[B(0)]^2\} = 0$ *it follows that* $B(0) = 0$.

$\mathrm{cov}(B(1), B(2)-B(1)) = \mathrm{cov}(B(1), B(2)) - \mathrm{cov}(B(1), B(1)) = \min(1,2) - \min(1,1) = 1 - 1 = 0$

Since $\mathrm{cov}(B(1), B(2) - B(1)) = 0$ *it follows that* $B(1)$ *and* $B(2) - B(1)$ *are uncorrelated, and since they are Gaussian, are thus independent. So,*

$$\begin{aligned} E[B(2)|B(1) = 0.5] &= E[B(1) + \{B(2) - B(1)\}|B(1) = 0.5] \\ &= E[B(1)|B(1) = 0.5] + E[B(2) - B(1)|B(1) = 0.5] \\ &= 0.5 + E[B(2) - B(1)] = 0.5. \end{aligned}$$

4. (20 marks)
Consider an up and in barrier call option that expires in $T = 4$ months, and that is monitored monthly. Assume that the stock price is initially $S(0) = \$100$ and satisfies a geometric Brownian motion with an interest rate of $r = 0\%$, and a volatility of 50%. Assume that the option strike price is $K = \$100$ and that the barrier is $B = \$120$. Your standard Gaussian random number generator gives you the following output:

$$X_1 = 0.5377, \quad X_2 = 1.8339, \quad X_3 = -2.2588, \quad X_4 = 0.8622, \ldots$$

Generate one stock price path and evaluate its payoff for this option.

Answer: The stock price path, $S(t)$ *for* $t = 1/12, 2/12, 3/12,$ *and* $4/12$ *years is given by*

$$\begin{aligned} S(1/12) &= S(0)e^{(-\sigma^2/2)(1/12)+\sigma\sqrt{1/12}X_1} = 100.0e^{-0.125/12+0.5\sqrt{1/12}(0.5377)} = 106.9 \\ S(2/12) &= S(1/12)e^{(-\sigma^2/2)(1/12)+\sigma\sqrt{1/12}X_2} = 106.9e^{-0.125/12+0.5\sqrt{1/12}(1.8339)} = 137.9 \\ S(3/12) &= S(2/12)e^{(-\sigma^2/2)(1/12)+\sigma\sqrt{1/12}X_3} = 137.9e^{-0.125/12+0.5\sqrt{1/12}(-2.2588)} = 98.5 \\ S(4/12) &= S(3/12)e^{(-\sigma^2/2)(1/12)+\sigma\sqrt{1/12}X_4} = 98.5e^{-0.125/12+0.5\sqrt{1/12}(0.8622)} = 110.4. \end{aligned}$$

Thus, the stock path exceeds the barrier at $t = 2$ *months and finally yields a payoff of* $\$110.4 - \$100 = \$10.4$.