MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test consists of FOUR questions. Answer all of them.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text. Computers are also allowed, but only using MATLAB, Mathematica, or JMP. No internet access.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit. Write out pseudo-code for the programs that you run to get your answers.*

1. (25 marks)
Let $Y_1, \ldots, Y_n$ be IID random variables with unknown mean, μ, and variance, σ^2. Consider the weighted Monte Carlo estimator for μ of the form

$$\hat{\mu} = w_1 Y_1 + \cdots + w_n Y_n,$$

where the w_i are fixed, non-negative numbers satisfying $w_1 + \cdots + w_n = 1$.

a) Show that $\hat{\mu}$ is unbiased, regardless of the choice of the w_i.

Answer: The unbiasedness of $\hat{\mu}$ follows because

$$\mathbb{E}(\hat{\mu}) = \mathbb{E}(w_1 Y_1 + \cdots + w_n Y_n) = w_1 \mathbb{E}(Y_1) + \cdots + w_n \mathbb{E}(Y_n) = (w_1 + \cdots + w_n)\mu = \mu.$$

b) Derive $\text{var}(\hat{\mu})$, the variance of $\hat{\mu}$, in terms of the w_i, μ, σ, and n.

Answer: The variance of $\hat{\mu}$ is

$$\text{var}(\hat{\mu}) = \text{var}(w_1 Y_1 + \cdots + w_n Y_n) = w_1^2 \,\text{var}(Y_1) + \cdots + w_n^2 \,\text{var}(Y_n) = (w_1^2 + \cdots + w_n^2)\sigma^2.$$

c) For the case $n = 2$ show that $\text{var}(\hat{\mu})$ is minimized when $w_1 = w_2 = 1/2$.

Answer: Since $w_1 + w_2 = 1$, it follows that $w_2 = 1 - w_1$, and so the variance of $\hat{\mu}$ is

$$\text{var}(\hat{\mu}) = \sigma^2[w_1^2 + (1 - w_1)^2] = \sigma^2[2w_1^2 - 2w_1 + 1] = 2\sigma^2[(w_1 - 1/2)^2 + 1/4].$$

This quantity has a minimum of $1/4$ when $w_1 = w_2 = 1/2$.

2. (25 marks)
Let Y be a random variable with a Pareto distribution. Specifically, the probability density function of Y is

$$f(y) = \begin{cases} \frac{4}{y^5}, & 1 \le y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Use the uniform pseudo-random numbers on $[0, 1]$,

$$X_i = 0.6312, 0.3551, 0.9970, 0.2242, 0.6525, 0.6050, 0.3872, 0.1422, \ldots,$$

to generate three IID Pareto pseudo-random numbers Y_1, Y_2, and Y_3.

Answer: The cumulative distribution of Y is

$$F(y) = \int_1^y f_Y(t)\,\mathrm{d}t = \int_1^y \frac{4}{t^5}\,\mathrm{d}t = \left.\frac{-1}{t^4}\right|_1^y = 1 - \frac{1}{y^4}$$

One may use the inverse transform method. Note that $F^{-1}(x) = 1/(1-x)^{1/4}$. Thus,

$$Y_i = F^{-1}(X_i) = \frac{1}{(1-X_i)^{1/4}} = 1.2832, 1.1159, 4.2740\ldots$$

or since $1 - X_i$ is also uniform,

$$Y_i = F^{-1}(1 - X_i) = \frac{1}{X_i^{1/4}} = 1.1219, 1.2954, 1.0008, \ldots.$$

3. (25 marks)
Each week the three gas stations near your home sell gas at a price per gallon of X_1, X_2 and X_3 respectively, where the X_i are IID uniform random variables on $[3.80, 4.20]$. You purchase gas at the station with the lowest price. Use Monte Carlo simulation to determine the average price per gallon of gas that you will pay to within the nearest penny and with 99% confidence.

Answer: You want to compute $\mu = \mathbb{E}(Y)$ where $Y = \min(X_1, X_2, X_3)$. The MATLAB code below does so. The sample size is found by trial and error. The average gas price that you pay is $\$3.91 \pm \$0.01$.

```
n0=1e4; %initial sample size
tol=0.01; %error tolerance
x=3.8+0.4*rand(n0,3); %uniform on [3.8,4.2]
price=min(x,[],2); %minimize over three columns
stdprice=std(price); %standard deviation of price you pay
nmu=ceil((2.58*1.2*stdprice/tol)^2) %needed sample size
x=3.8+0.4*rand(nmu,3); %uniform on [3.8,4.2]
price=min(x,[],2); %minimize over three columns
avgprice=mean(price) %sample average price you pay
stdprice=std(price); %standard deviation of price you pay
errorprice=2.58*stdprice/sqrt(nmu) %error of average price you pay
```

```
nmu =
   563
avgprice =
   3.9000e+00
errorprice =
   8.1967e-03
```

4. (25 marks)
   Consider two portfolios that each start with \$10,000 and whose values one year later (in thousands of dollars), V_1 and V_2, are given by

$$V_1 = 10\mathrm{e}^{-0.125+0.5X_1} \quad \text{and} \quad V_2 = 10\mathrm{e}^{-0.32+0.8X_2}, \qquad X_1, X_2 \text{ IID } \mathcal{N}(0,1),$$

   respectively. Use Monte Carlo simulation to compute the probability that the first portfolio outperforms the second. Your answer should be correct to the nearest 1% and with 99% confidence.

   Answer: The following MATLAB code computes the probability to be $58\% \pm 1\%$. *Note that the average values of the portfolios will be the same, but the first is more conservative.*

```
n0=1e4;%initial sample size
tol=0.01; %error tolerance
%value of two portfolios
value=10*exp([-0.125 + 0.5*randn(n0,1) -0.32 + 0.8*randn(n0,1)]);
yes=value(:,1)>value(:,2); %true if first one beats the second one
avgbeats=mean(yes); %sample proportion
stdbeats=sqrt(avgbeats*(1-avgbeats)); %estimated standard deviation
nmu=ceil((2.58*1.2*stdbeats/tol)^2) %needed sample size
%value of two portfolios
value=10*exp([-0.125 + 0.5*randn(nmu,1) -0.32 + 0.8*randn(nmu,1)]);
yes=value(:,1)>value(:,2); %true if first one beats the second one
avgbeats=mean(yes) %sample proportion
stdbeats=sqrt(avgbeats*(1-avgbeats)); %estimated standard deviation
errorbeats=2.58*stdbeats/sqrt(nmu) %error in probability of first beating second

nmu =
       23264
avgbeats =
   5.7956e-01
errorbeats =
   8.3498e-03
```