MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test consists of THREE questions. Answer all of them.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text. No internet access.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit. Write out pseudo-code for the programs that you run to get your answers.*

1. (30 points)
Suppose that Y is a random variable with finite mean, μ, and finite standard deviation, $\sigma > 0$. Let $Y_1, Y_2, \ldots$ be IID instances of Y, and let

$$\overline{Y}_n = \frac{1}{n}(Y_1 + \cdots + Y_n)$$

be the sample mean of the first n samples.

a) Verify that $\overline{Y}_n$ is unbiased.

Answer:

$$\mathbb{E}(\overline{Y}_n) = \mathbb{E}\left[\frac{1}{n}(Y_1 + \cdots + Y_n)\right] = \frac{1}{n}\left[\mathbb{E}(Y_1) + \cdots + \mathbb{E}(Y_n)\right] = \frac{1}{n}\left[\mu + \cdots + \mu\right] = \mu$$

b) Next compute the variance of $\overline{Y}_n$.

Answer:

$$\begin{aligned}\operatorname{var}(\overline{Y}_n) &= \operatorname{var}\left(\frac{1}{n}(Y_1 + \cdots + Y_n)\right) = \frac{1}{n^2}[\operatorname{var}(Y_1) + \cdots + \operatorname{var}(Y_n)] \\ &= \frac{1}{n^2}(\sigma^2 + \cdots + \sigma^2) = \frac{\sigma^2}{n}\end{aligned}$$

c) Now consider the estimator $\overline{Y}_N$, where the number of samples, N, is a *random* variable that is independent of the Y_i with probability mass function

$$\mathbb{P}(N = n) = p_n, \qquad n = 1, 2, \ldots,$$

for some known p_n. The laws of conditional probability say that

$$\mathbb{E}(\overline{Y}_N) = \mathbb{E}[\mathbb{E}(\overline{Y}_N|N)], \qquad \operatorname{var}(\overline{Y}_N) = \mathbb{E}[\operatorname{var}(\overline{Y}_N|N)] + \operatorname{var}[\mathbb{E}(\overline{Y}_N|N)].$$

In part a) you already computed $\mathbb{E}(\overline{Y}_N|N = n) = \mathbb{E}(\overline{Y}_n)$, and in part b) you already computed $\operatorname{var}(\overline{Y}_N|N = n) = \operatorname{var}(\overline{Y}_n)$. Use these results to determine whether $\overline{Y}_N$ is unbiased and also to determine $\operatorname{var}(\overline{Y}_N)$ in terms of the p_n above.

Answer: Since $\mathbb{E}(\overline{Y}_N|N=n)=\mu$*, independent of* n*, it follows that* $\mathbb{E}(\overline{Y}_N|N)=\mu$*, and so*

$$\mathbb{E}(\overline{Y}_N)=\mathbb{E}[\mathbb{E}(\overline{Y}_N|N)]=\mathbb{E}[\mu]=\mu,$$

so $\overline{Y}_N$ *is unbiased. Also,* $\mathrm{var}(\overline{Y}_N|N=n)=\sigma^2/n$*, and so* $\mathrm{var}(\overline{Y}_N|N)=\sigma^2/N$*, and*

$$\begin{aligned}\mathrm{var}(\overline{Y}_N)&=\mathbb{E}[\mathrm{var}(\overline{Y}_N|N)]+\mathrm{var}[\mathbb{E}(\overline{Y}_N|N)]\\&=\mathbb{E}\left[\frac{\sigma^2}{N}\right]+\mathrm{var}(\mu)\\&=\sum_{n=1}^{\infty}\frac{\sigma^2}{n}p_n+0=\sigma^2\sum_{n=1}^{\infty}\frac{p_n}{n}.\end{aligned}$$

2. (30 points)
Suppose that Y is a random variable with the probability density function ρ and cumulative distribution function, F, given by

$$\rho(y)=\frac{\mathrm{e}^{-y}}{1-\mathrm{e}^{-1}},\quad F(y)=\frac{1-\mathrm{e}^{-y}}{1-\mathrm{e}^{-1}},\qquad 0\le y\le 1.$$

Given the *uniform* pseudorandom numbers

i	1	2	3	4
X_i	0.27850	0.54688	0.95751	0.96489

construct *one* pseudo-random number Y_1 by *either* the inverse distribution transformation method or acceptance-rejection sampling.

Answer: For the inverse distribution transformation method we first construct the inverse distribution (or quantile) function:

$$x=F(y)=\frac{1-\mathrm{e}^{-y}}{1-\mathrm{e}^{-1}}\iff(1-\mathrm{e}^{-1})x=1-\mathrm{e}^{-y}\iff \mathrm{e}^{-y}=1-(1-\mathrm{e}^{-1})x$$
$$\iff y=-\log(1-[1-\mathrm{e}^{-1}]x)=:F^{-1}(x).$$

Taking X_1 *above we get*

$$Y_1=-\log(1-[1-\mathrm{e}^{-1}]X_1)=-\log(1-[1-\mathrm{e}^{-1}]0.27850)=0.19364$$

(Remember that log *means* natural *log or* ln *on your calculator.)*

For acceptance-rejection sampling, we let ρ_u *denote the density function of a* $\mathcal{U}[0,1]$ *random variable, i.e.,* $\rho_u(x)=1$*, and note that* $(1-\mathrm{e}^{-1})\rho(y)=\mathrm{e}^{-y}\le\rho_u(y)$ *for* $0\le y\le 1$*. First we take* $X_1=0.27850$*, then we check whether*

$$0.54688=X_2\le(1-\mathrm{e}^{-1})\rho(X_1)=\mathrm{e}^{-X_1}=\mathrm{e}^{-0.27850}=0.75692.$$

Since this is true, we accept 0.27850 *as* Y_1*.*

3. (40 points)
Consider the situation of $r = 0$ interest rate. A stock is modeled by a geometric Brownian motion and has an initial price of \$50 and a volatility of 40% (for the whole year). An Asian arithmetic mean call option monitored weekly expires in four weeks. (You may assume 52 weeks per year.) The strike price is \$48.

a) Given the $\mathcal{N}(0,1)$ pseudo-random numbers

$$0.71724, -0.35225, -0.37983, -1.76323, \ldots$$

construct a single stock path and the Asian arithmetic mean call payoff for this path.

Answer: The geometric Brownian motion for the stock price is

$$\begin{aligned}
t_j &= \frac{j}{52}, \qquad j = 0, \ldots, 4 \\
B(0) &= 0 \\
B(t_j) &= B(t_{j-1}) + \sqrt{1/52} X_j, \quad j = 0, \ldots, 4, \qquad X_1, X_2, \ldots \sim \mathcal{N}(0,1), \\
S(t_j) &= S(0)\exp((r - \sigma^2/2)t_j + \sigma B(t_j)), \quad j = 0, \ldots, 4 \\
&= 50\exp(-0.08 t_j + 0.4 B(t_j))
\end{aligned}$$

j	0	1	2	3	4
t_j	0	1/52	1/26	3/52	1/13
X_j		0.71724	−0.35225	−0.37983	−1.76323
$B(t_j)$	0.00000	0.09946	0.05061	−0.00206	−0.24657
$S(t_j)$	50.00	51.95	50.87	49.73	45.03

Thus the payoff is

$$\max\left(\frac{1}{4}\sum_{j=1}^{4} S(t_j) - K, 0\right) = \max(49.39 - 48, 0) = \$1.39.$$

b) Suppose that the sample mean and sample standard deviation of $n = 1000$ payoffs are \$2.71 and \$2.93, respectively. Using an approximate central limit theorem (CLT) confidence interval, determine the sample size needed to estimate the option price with an error or no more than \$0.01 and a confidence level of 99%.

Answer: The half-width of the CLT confidence interval is

$$\frac{2.58\sigma}{\sqrt{n}} \lessapprox \frac{2.58 \times 1.2 \times 2.93}{\sqrt{n}} = \frac{9.10}{\sqrt{n}}.$$

Thus we choose $n \approx (9.10/0.01)^2 \approx 830{,}000$.