MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. This test has THREE questions for a total of 100 *points possible. You should attempt them all.*

ii. The time allowed is 75 minutes.

iii. This test is closed book, but you may use 4 double-sided letter-size sheets of notes.

iv. (Programmable) calculators are allowed, but they must not have stored text.

v. Show all your work to justify your answers. Answers without adequate justification will not receive credit.

The following are independent and identically distributed (IID) random variables from two different distributions. You will use some of these to solve some of the problems below.

Standard uniform, $\mathcal{U}[0,1]$	0.5173	0.9470	0.7655	0.2824	0.2210	0.6862
Standard normal, $\mathcal{N}(0,1)$	0.5335	−0.2491	−2.0345	3.0037	0.1657	0.2510

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1. (40 points)
Let Y be a random variable with unknown mean μ and variance σ^2. Let

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i, \quad \hat{\sigma}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \hat{\mu}_n)^2, \qquad Y_1, Y_2, \ldots \overset{\text{IID}}{\sim} Y.$$

a) What are $\mathbb{E}(\hat{\mu}_n)$ and $\operatorname{var}(\hat{\mu}_n)$ in terms of μ, σ, and n?

Answer:

$$\mathbb{E}(\hat{\mu}_n) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(Y_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu,$$

$$\operatorname{var}(\hat{\mu}_n) = \operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{var}(Y_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}$$

b) Consider the statement: "$\hat{\mu}_{20\,000}$ is 100 times closer to μ than $\hat{\mu}_{200}$ is". Give at least one reason why this statement is false. Give a mathematically precise correction to this statement.

Answer: We cannot compare two random variables except in some probabilistic sense. Define the root mean squared error of $\hat{\mu}_n$ as

$$\operatorname{rmse}(\hat{\mu}_n) := \sqrt{\mathbb{E}[(\hat{\mu}_n - \mu)^2]}.$$

Note that since $\hat{\mu}_n$ is an unbiased estimator,

$$\operatorname{rmse}(\hat{\mu}_n) = \sqrt{\operatorname{var}(\hat{\mu}_n)} = \frac{\sigma}{\sqrt{n}}.$$

Thus,

$$\operatorname{rmse}(\hat{\mu}_{20\,000}) = \frac{\sigma}{\sqrt{20\,000}} = \frac{\sigma}{100\sqrt{2}}, \quad \operatorname{rmse}(\hat{\mu}_{200}) = \frac{\sigma}{\sqrt{200}} = \frac{\sigma}{10\sqrt{2}},$$

and $\hat{\mu}_{20\,000}$ is 10 *(not* 100*) times closer to μ than $\hat{\mu}_{200}$ is, in the sense of root mean squared error.*

c) If $\hat{\sigma}^2_{1000} = 4.3$, how many samples should be used to estimate μ with an absolute error tolerance of 0.02 with a confidence level of 99% based on a Central Limit Theorem (CLT) approximation?

Answer: Choosing a standard deviation inflation factor we have CLT confidence intervals of half-width $2.58 \times 1.2 \times \sqrt{4.3}/\sqrt{n}$. *Setting this to be* 0.02, *we get*

$$n = \left\lceil \frac{(2.58)^2 \times (1.2)^2 \times 4.3}{(0.02)^2} \right\rceil = 103\,042$$

d) Again assume that $\hat{\sigma}^2_{1000} = 4.3$ as in part c), and suppose that $\hat{\mu}_n = 12.456$ for n chosen according to part c). If you compute another $\hat{\mu}_n$ with the same n, but for a sample independent of the previous one, would you expect it to be 12.456 also? Explain why. Would you be surprised if this new $\hat{\mu}_n$ were 13.456? Explain why.

Answer: We expect that independent $\hat{\mu}_n$ *with the same* n *should be somewhat less than* ± 0.02 *away from the original. Since* $\hat{\mu}_n$ *is random, we would not expect two different* $\hat{\mu}_n$ *to agree to three digits, and we would be surprised if the new* $\hat{\mu}_n$ *differed by more than* 0.04 *from the old one. Thus our answers are "no" and "yes", respectively.*

e) If $\hat{\sigma}^2_{1000} = 4.3$, what would you expect $\hat{\sigma}^2_{100\,000}$ to be, roughly speaking?

Answer: Since $\hat{\sigma}^2_n$ *is an unbiased estimator of* σ^2 *for all* n, *and both* n *are relatively large, and* $\hat{\sigma}^2_{100\,000}$ *should be about the same as* $\hat{\sigma}^2_{1000}$, *i.e.,* 4.3.

2. (30 points)
Let Y have a probability density function (PDF) of ϱ, a particular Pareto distribution, namely,

$$\varrho(y) = \begin{cases} 0, & -\infty < y < 1, \\ \dfrac{a}{y^5}, & 1 \le y < \infty, \end{cases}$$

a) What should the value of a be?

Answer: The corresponding cumulative distribution function (CDF) is

$$F(y) = \int_1^y \frac{a}{x^5}\,\mathrm{d}x = \frac{-a}{4x^4}\bigg|_1^y = \frac{a}{4}\left(1 - \frac{1}{y^4}\right).$$

Since $1 = \lim_{y\to\infty} F(y) = a/4$, *we need to choose* $a = 4$. *Thus,* $a = 4$.

b) What is the corresponding cumulative distribution function (CDF)?

Answer: From above

$$F(y) = 1 - \frac{1}{y^4}.$$

c) Use the random numbers at the beginning of this test to generate *two* IID values of Y.

Answer: We use the inverse CDF transformation:

$$x = F(y) = 1 - \frac{1}{y^4} \implies y = \frac{1}{(1-x)^{1/4}} = F^{-1}(x)$$

$X \sim \mathcal{U}[0,1]$	0.5173	0.9470
$Y = F^{-1}(X) = \dfrac{1}{(1-X)^{1/4}} \sim F$	1.1997	2.0842
or since $1-X$ *is also* $\mathcal{U}[0,1]$, $Y = F^{-1}(1-X) = \dfrac{1}{X^{1/4}} \sim F$	1.1791	1.0137

3. (30 points)
Consider a stock whose price is modeled by a geometric Brownian motion. The price today is \$30, the volatility is 40% year$^{-1/2}$, and the risk-free interest rate is 1.5% per year.

a) Compute one stock path at times of 1, 2, and 3 *months* from now.

Answer: Since

$$S(t) = S(0)\exp((r - \sigma^2/2)t + \sigma B(t)) = 30\exp((0.015 - 0.4^2/2)t + 0.4B(t))$$
$$= 30\exp(-0.065t + 0.4B(t))$$

we first need to compute a Brownian motion, B at the three times (in years):

$$B(0) = 0, \qquad B(j/12) = B((j-1)/12) + \sqrt{1/12}Z_j, \qquad Z_j \overset{\text{IID}}{\sim} \mathcal{N}(0,1),\ j = 1,2,3.$$

We get

j	1	2	3
t_j	1/12	1/6	1/4
Z_j	0.5335	−0.2491	−2.0345
$B(t_j)$	$0 + \sqrt{1/12}Z_1$ 0.1540	$B(t_1) + \sqrt{1/12}Z_2$ 0.0821	$B(t_2) + \sqrt{1/12}Z_3$ −0.5052
$S(t_j)$	$30\exp(-0.065(1/12)$ $+0.4B(1/12))$ 31.7339	$30\exp(-0.065(1/6)$ $+0.4B(1/6))$ 30.6674	$30\exp(-0.065(1/4)$ $+0.4B(1/4))$ 24.1157

b) Based on your stock price path in the previous part of the problem, what is the discounted payoff of a European put option with a strike price of \$25 and an expiration date of 2 months from now? What is the payoff if the expiration date is 3 months from now?

Answer: For $T = 1/6$ year or 2 months, we have

$$\textit{discounted payoff} = \max(K - S(T), 0)\mathrm{e}^{-rT} = \max(25 - 30.6674, 0)\mathrm{e}^{-0.015(1/6)} = 0.$$

For $T = 1/4$ year or 3 months, we have

$$\textit{discounted payoff} = \max(K - S(T), 0)\mathrm{e}^{-rT} = \max(25 - 24.1157, 0)\mathrm{e}^{-0.015(1/4)} = 0.8810.$$