MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test has THREE questions for a total of* 100 *points possible. You should attempt them all.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (33 points)
Let Y be a random variable with mean a, variance a^2, $\mathbb{E}(Y^3) = 4a^3$, and $\mathbb{E}(Y^4) = 10a^4$. The value of a is *unknown*. The task is to estimate $M = \mathbb{E}(Y^2)$. Let $Y_1, Y_2, \ldots \overset{\text{IID}}{\sim} Y$.

a) Is $W := \left(\frac{1}{n}\sum_{i=1}^n Y_i\right)^2$ a biased or unbiased estimator for M? Is W an asymptotically unbiased estimator as $n \to \infty$?

Answer: Note that $M = \mathbb{E}(Y^2) = \text{var}(Y) + [\mathbb{E}(Y)]^2 = 2a^2$.

$$\mathbb{E}(W) = \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n Y_i\right)^2\right] = \frac{1}{n^2}\sum_{i,j=1}^n \mathbb{E}(Y_iY_j) = \frac{1}{n^2}\bigl\{n(n-1)[\mathbb{E}(Y)]^2 + n\,\mathbb{E}(Y^2)\bigr\}$$
$$= \frac{1}{n^2}\bigl\{n(n-1)a^2 + n2a^2\bigr\} = \frac{a^2}{n}\bigl\{n-1+2\bigr\} = \frac{n+1}{n}a^2 \begin{cases} \neq 2a^2 & \text{for } n > 1, \\ \not\to 2a^2 & \text{as } n \to \infty. \end{cases}$$

Thus, W is a biased estimator, unless $n = 1$, and it is an asymptotically biased estimator.

b) Is $V := \frac{1}{n}\sum_{i=1}^n Y_i^2$ a biased or unbiased estimator for M? What is the variance of V?

Answer:

$$\mathbb{E}(V) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n Y_i^2\right) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}(Y_i^2) = \frac{1}{n}n\,\mathbb{E}(Y^2) = 2a^2 \quad \textit{so } V \textit{ is unbiased},$$
$$\text{var}(V) = \text{var}\left(\frac{1}{n}\sum_{i=1}^n Y_i^2\right) = \frac{1}{n^2}\sum_{i=1}^n \text{var}(Y_i^2) = \frac{\text{var}(Y^2)}{n} = \frac{\mathbb{E}(Y^4) - [\mathbb{E}(Y^2)]^2}{n}$$
$$= \frac{10a^4 - (2a^2)^2}{n} = \frac{6a^4}{n}.$$

2. (33 points)
Consider the following probability density function (PDF) for the random variable Y with sample space $[0,1]$:
$$\varrho(y) = k[1 - \cos(2\pi y)].$$

a) What is the value of k?

Answer:

$$1 = \int_0^1 \varrho(y) = k \int_0^1 [1 - \cos(2\pi y)]\, \mathrm{d}y = k \left[y - \frac{\sin(2\pi y)}{2\pi} \right]\Bigg|_0^1 = k,$$

so $k = 1$.

b) Suppose that you want to generate instances of Y using instances of $X \sim \mathcal{U}[0,1]$ and *acceptance-rejection* sampling. On average, what percentage of X values will you accept if you arrange to accept as many as possible.

Answer: We need to find the largest c such that $\varrho(y) \le 1/c$ *for* $y \in [0,1]$*, since* 1 *is the PDF for the uniform distribution. Since*

$$\max_{0 \le y \le 1} \varrho(y) = \max_{0 \le y \le 1} [1 - \cos(2\pi y)] = 2 = 1 - \cos(\pi) = \varrho(1/2),$$

the best c possible is $c = 1/2$*. This means that half of the X values generated will be accepted on average.*

c) If $X = 1/3$, what is the chance that it will be accepted?

Answer: The chance that X will be accepted is $c\varrho(X)$*, which in this case is* $0.5\varrho(1/3) = 0.5[1 - \cos(2\pi/3)] = 0.5(1 + 0.5) = 0.75$.

3. (34 points)
Consider a stock whose price is modeled by a geometric Brownian motion. The price today is \$50, the volatility is 50% year$^{-1/2}$, and the risk-free interest rate is 1% per year. The following are standard normal (Gaussian) independent and identically distributed (IID) random variables:

$$0.1827 \quad -0.7924 \quad -0.2972 \quad 0.6409$$

a) Compute *one* stock path at times of 1, 2, and 3 *months* from now.

Answer: Since

$$\begin{aligned} S(t) = S(0)\exp((r - \sigma^2/2)t + \sigma B(t)) &= 50 \exp((0.01 - 0.5^2/2)t + 0.5B(t)) \\ &= 50 \exp(-0.115t + 0.5B(t)) \end{aligned}$$

we first need to compute a Brownian motion, B at the three times (in years):

$$B(0) = 0, \qquad B(j/12) = B((j-1)/12) + \sqrt{1/12} Z_j, \qquad Z_j \overset{\text{IID}}{\sim} \mathcal{N}(0,1), \; j = 1, 2, 3.$$

We get

j	0	1	2	3
t_j	0	1/12	1/6	1/4
Z_j		0.1827	−0.7924	−0.2972
$B(t_j)$	0	$0 + \sqrt{1/12}Z_1$ 0.0527	$B(t_1) + \sqrt{1/12}Z_2$ −0.1760	$B(t_2) + \sqrt{1/12}Z_3$ −0.2618
$S(t_j)$	50	$50\exp((-0.115(1/12)$ $+0.5B(1/12))$ 50.8464	$50\exp((-0.115(1/6)$ $+0.5B(1/6))$ 44.9187	$50\exp((-0.115(1/4)$ $+0.5B(1/4))$ 42.6221

b) Based on your stock price path in the previous part of the problem, what is the discounted payoff of a lookback put option with an expiration date of 3 months from now?

Answer: For $T = 1/4$ year or 3 months, we have

$$\textit{discounted payoff} = \left[\max_{t=0,1/12,1/6,1/4} S(t) - S(T)\right] \mathrm{e}^{-rT}$$
$$= [50.8464 - 42.6221]\mathrm{e}^{-0.01(1/4)} = 8.2038$$

c) You find that the sample variance of 10 000 IID discounted payoffs is 17. How many paths should you use to estimate the option price to the nearest penny ($0.01)?

Answer: Using the Central Limit Theorem, we want

$$\frac{2.58\sigma}{\sqrt{n}} \le \frac{2.58 \times 1.2\hat{\sigma}}{\sqrt{n}} \le 0.01 = \textit{penny} \implies n \ge \left\lceil \left(\frac{2.58 \times 1.2\sqrt{17}}{0.01}\right)^2 \right\rceil \approx 1.63 \times 10^6.$$