MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test has FOUR questions. Attempt them all. The maximum number of points is* 100.

ii. *The time allowed is 75 minutes.*

iii. *Keep at least* four *significant digits in your intermediate calculations and final answers.*

iv. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

v. *(Programmable) calculators are allowed, but they must not have stored text.*

vi. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (30 points)
A complicated simulation code produces a sequence of IID Bernoulli random variables $Y_1, Y_2, \ldots$, where $\mathbb{E}(Y_i) = p$ and $\mathrm{var}(Y_i) = p(1-p)$ for all i. However, p is unknown. Let

$$\widehat{P}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

a) Demonstrate that $\widehat{P}_n$ is an *unbiased* estimator for p.

Answer:

$$\mathbb{E}(\widehat{P}_n) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(Y_i) = \frac{1}{n}\sum_{i=1}^{n} p = p$$

b) What is the bias of $\widehat{V}_n = \widehat{P}_n(1-\widehat{P}_n)$ as an estimator for $p(1-p)$, the variance of the Y_i? What is the limiting value of the bias as $n \to \infty$? What is the limiting value of $\mathbb{E}(\widehat{V}_n)$ as $n \to \infty$?

Answer:

$$\begin{aligned}
\mathrm{bias}(\widehat{V}_n) &= p(1-p) - \mathbb{E}(\widehat{V}_n) = p(1-p) - \mathbb{E}[\widehat{P}_n(1-\widehat{P}_n)] = p(1-p) - \mathbb{E}(\widehat{P}_n) + \mathbb{E}(\widehat{P}_n^2) \\
&= -p^2 + \mathbb{E}\left(\frac{1}{n^2}\sum_{i,j=1}^{n} Y_i Y_j\right) = -p^2 + \frac{1}{n^2}\sum_{i,j=1}^{n}\mathbb{E}(Y_i Y_j).
\end{aligned}$$

Now $\mathbb{E}(Y_i^2) = \mathbb{E}(Y_i) = p$, *and for* $i \ne j$, $\mathbb{E}(Y_i Y_j) = \mathbb{E}(Y_i)\,\mathbb{E}(Y_j) = p^2$. *Thus*

$$\mathrm{bias}(\widehat{V}_n) = -p^2 + \frac{np + (n^2-n)p^2}{n^2} = \frac{p(1-p)}{n}.$$

As $n \to \infty$, *the bias goes to zero, and* $\widehat{V}_n$ *goes to* $p(1-p)$.

c) What is $\mathrm{var}(\widehat{P}_n)$ as a function of p and n? What is the limiting value of $\mathrm{var}(\widehat{P}_n)$ as $n \to \infty$?

Answer:

$$\mathrm{var}(\widehat{P}_n) = \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}(Y_i) = \frac{\mathrm{var}(Y_1)}{n} = \frac{p(1-p)}{n}$$

As $n \to \infty$, $\mathrm{var}(\widehat{P}_n) \to 0$.

2. (30 points)
Consider the situation in problem where after generating an IID sample of 1000 instances of Y_i, you find that 434 are 1 (success) and the remainder are 0 (failure).

a) Using the Central Limit Theorem construct an approximate 99% confidence interval for p, the probability of success, based on this sample.

Answer: The Central Limit Theorem says that

$$\Pr\left[\left|p - \widehat{P}_n\right| \le \frac{2.58\sigma}{\sqrt{n}}\right] \gtrapprox 99\%$$

Note that $\hat{p}_{1000} = 0.434$ *in this case. To ensure that we do not under estimate* σ*, we replace it by* $\hat{\sigma} = 1.2\sqrt{\hat{v}_{1000}} = 1.2\sqrt{0.434(1-0.434)} = 0.5947$. *The half-width of the* 99% *approximate confidence interval is*

$$\frac{2.58\hat{\sigma}}{\sqrt{1000}} = \frac{2.58 \times 0.5947}{\sqrt{1000}} = 0.04852.$$

The confidence interval is $0.434 \pm 0.04852 = [0.3855, 0.4825]$.

b) If you want a 99% confidence interval for p with half-width of 0.002, how large should n be?

Answer: Recognizing that the width of the confidence interval is

$$\frac{2.58\hat{\sigma}}{\sqrt{n}} = \frac{2.58 \times 0.5947}{\sqrt{n}} = \frac{1.534}{\sqrt{n}},$$

it follows that we need

$$n \ge \left\lceil \left(\frac{1.534}{0.002}\right)^2 \right\rceil \approx 588600$$

3. (30 points)
Consider the following three dimensional integral

$$\mu = \int_{[-1,1]^3} \cos\left(\sqrt{x_1^2 + x_2^2} + x_3\right)\, \mathrm{d}\boldsymbol{x}.$$

The following are six IID $\mathcal{U}[0,1]$ random numbers:

$$0.1135 \quad 0.9745 \quad 0.7287 \quad 0.3515 \quad 0.7076 \quad 0.7996$$

Use these to form, $\hat{\mu}_n$, a Monte Carlo estimate of μ. Granted, n cannot be very large given only six IID $\mathcal{U}[0,1]$ random numbers; how large can your n be?

Answer: Since our domain is $[-1,1]^3$, we must write the integral as

$$\mu = \int_{[-1,1]^3} \underbrace{8\cos\left(\sqrt{x_1^2+x_2^2}+x_3\right)}_{f(\boldsymbol{x})} \underbrace{\frac{1}{8}}_{\varrho(\boldsymbol{x})} \,\mathrm{d}\boldsymbol{x}.$$

Here the probability density function is constant, corresponding to the uniform density on the domain.

We must transform our $\mathcal{U}[0,1]$ random numbers to $\mathcal{U}[-1,1]$ random numbers:

i	1	2	3	4	5	6
$U_i \sim \mathcal{U}[0,1]$	0.1135	0.9745	0.7287	0.3515	0.7076	0.7996
$Z_i = 2U_i - 1 \sim \mathcal{U}[-1,1]$	−0.7730	0.9490	0.4575	−0.2971	0.4152	0.5992

Next, we order the six Z_i to make two vectors in $[-1,1]^3$, i.e., $n=2$. Depending on the way we do the ordering there are two ways to do this. Our Monte Carlo estimate is then the average of the two function values. The first way is

i		$\boldsymbol{x}_i$		$f(\boldsymbol{x}_i)$
1	−0.7730	0.9490	0.4575	−0.8833
2	−0.2971	0.4152	0.5992	3.5591
$\hat{\mu}_2$				1.338

The other way is

i		$\boldsymbol{x}_i$		$f(\boldsymbol{x}_i)$
1	−0.7730	0.4575	0.4152	2.0361
2	0.9490	−0.2971	0.5992	−0.1823
$\hat{\mu}_2$				0.9269

4. (10 points)
Let B denote a Brownian motion. By hand compute, $D(t)$, the variance of a difference quotient:

$$D(t) = \operatorname{var}\left(\frac{B(t)-B(0)}{t}\right), \qquad t > 0.$$

You may use the assumed properties of a Brownian motion. What is the limiting value of $D(t)$ as $t \downarrow 0$?

Answer: We know that $B(t)-B(0) \sim \mathcal{N}(0,t)$, which means that

$$D(t) = \operatorname{var}\left(\frac{B(t)-B(0)}{t}\right) = \frac{\operatorname{var}\left(B(t)-B(0)\right)}{t^2} = \frac{t}{t^2} = \frac{1}{t}.$$

Thus, as $t \downarrow 0$, $D(t) \to \infty$. This explains why Brownian motions have no derivative with probability one.