MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test has FOUR questions. Attempt them all. The maximum number of points is* 100.

ii. *The time allowed is 75 minutes.*

iii. *Keep at least* four *significant digits in your intermediate calculations and final answers.*

iv. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

v. *(Programmable) calculators are allowed, but they must not have stored text. No phones are allowed.*

vi. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (15 points)
Consider the following estimators for $\mu = \mathbb{E}(Y)$, where the random variable Y has variance σ^2:

$$Z_1 = \frac{Y_1 + Y_2}{10}, \quad Z_2 = 10Y_1 - 9Y_2, \quad Z_3 = 47, \qquad \text{where } Y_1, Y_2 \overset{\text{IID}}{\sim} Y.$$

a) What are the *biases* of Z_1, Z_2, and Z_3? Which estimator(s) is unbiased?

Answer:

$$\begin{aligned}
\text{bias}(Z_1) &= \mu - \mathbb{E}(Z_1) = \mu - \frac{\mathbb{E}(Y_1) + \mathbb{E}(Y_2)}{10} = \mu - \frac{\mu + \mu}{10} = \frac{4\mu}{5}, \\
\text{bias}(Z_2) &= \mu - \mathbb{E}(Z_2) = \mu - [10\,\mathbb{E}(Y_1) - 9\,\mathbb{E}(Y_2)] = \mu - [10\mu - 9\mu] = 0 \qquad \textit{(unbiased)}, \\
\text{bias}(Z_3) &= \mu - \mathbb{E}(Z_3) = \mu - 47.
\end{aligned}$$

b) What are the *variances* of Z_1, Z_2, and Z_3? Which estimator has the smallest variance?

Answer:

$$\begin{aligned}
\text{var}(Z_1) &= \text{var}\left(\frac{Y_1 + Y_2}{10}\right) = \frac{\text{var}(Y_1) + \text{var}(Y_2)}{100} = \frac{\sigma^2}{50}, \\
\text{var}(Z_2) &= \text{var}(10Y_1 - 9Y_2) = 100\sigma^2 + 81\sigma^2 = 181\sigma^2, \\
\text{var}(Z_3) &= \text{var}(47) = 0 \qquad \textit{(smallest variance)}.
\end{aligned}$$

2. (30 points)
Suppose that you want to estimate $\mu = \mathbb{E}(Y)$ and that you can generate IID instances of Y.

a) Suppose that you an observe a sample mean of 7.532 and an (unbiased) sample variance of 34.73 from 1000 IID instances of Y. Construct an approximate 99% confidence interval for μ.

Answer: Using the Central Limit Theorem (CLT) and inflating the sample standard deviation by 1.2 *give an approximate confidence interval of*

$$\hat{\mu}_n \pm \frac{2.58 \times 1.2 \times \hat{\sigma}_n}{\sqrt{n}} = 7.532 \pm \frac{2.58 \times 1.2 \times \sqrt{34.73}}{\sqrt{1000}} = 7.532 \pm 0.5770 = [6.955, 8.109]$$

b) Suppose that, *independently* of part a), you observe a sample mean of 7.568 from 5000 IID instances of Y. Find the mean of this second sample *combined with* the first sample in part a).

Answer: The total number of data is now $1000 + 5000 = 6000$. *The sample mean of the both samples combined is*

$$\hat{\mu}_{6000} = \frac{1}{6000}\sum_{i=1}^{6000} Y_i = \frac{1}{6000}\left[\sum_{i=1}^{1000} Y_i + \sum_{i=1000+1}^{6000} Y_i\right] = \frac{1}{6000}\left[1000 \times 7.532 + 5000 \times 7.568\right]$$
$$= 7.562$$

c) What would you expect the half width of your approximate confidence interval based on the *combined samples* in part a) and b) to be?

Answer: Now $n = 1000 + 5000 = 6000$, *which is six times the size of the original sample. Thus, the half-width of the approximate confidence interval will be* $1/\sqrt{6}$ *that of original, or* $0.5770/\sqrt{6} = 0.2356$.

3. (30 points)
Consider the random variable X with the cumulative distribution function

$$F(x) = \begin{cases} 0, & -\infty < x < 0, \\ 1 - \dfrac{\exp(-x)}{2}, & 0 \le x < \infty. \end{cases}$$

a) What is $\mathbb{P}(X < 0)$? $\mathbb{P}(X = 0)$? $\mathbb{P}(X > 0)$?

Answer:

$$\mathbb{P}(X < 0) = \lim_{x \uparrow 0} \mathbb{P}(X \le x) = \lim_{x \uparrow 0} F(x) = 0,$$
$$\mathbb{P}(X = 0) = \mathbb{P}(X \le 0) - \mathbb{P}(X < 0) = F(0) - 0 = \frac{1}{2},$$
$$\mathbb{P}(X > 0) = 1 - \mathbb{P}(X \le 0) = 1 - F(0) = \frac{1}{2}.$$

b) Suppose that $Z \sim \mathcal{U}[0, 1]$. How can you obtain an instance of X from an instance of Z?

Answer: We can use the inverse CDF transformation. Note that X *has sample space* $[0, \infty)$. *Since* $\mathbb{P}(X = 0) = 1/2$, *then* $0 \le Z \le 1/2$ *corresponds to* $X = 0$. *For* $1/2 < Z < 1$, *we solve the equation*

$$Z = F(X) = 1 - \frac{\exp(-X)}{2} \iff \exp(-X) = 2(1 - Z) \iff X = -\log(2(1 - Z)).$$

To summarize

$$X = F^{-1}(Z) = \begin{cases} 0, & 0 \le Z \le \dfrac{1}{2}, \\ -\log(2(1 - Z)), & \dfrac{1}{2} < Z < 1. \end{cases}$$

c) What values of X are produced by your recipe in part b) for $Z_1 = 0.3281$ and $Z_2 = 0.6043$?

Answer:

$$X_1 = 0 \textit{ since } Z_1 = 0.3281 \le 1/2,$$
$$X_2 = -\log(2(1 - Z_2)) = -\log(2(1 - 0.6043)) = 0.2340 \textit{ since } Z_2 = 0.6043 > 1/2,$$

4. (25 points)
Consider the following two-dimensional integral:

$$\mu = \int_{\mathbb{R}^2} \cos(x_1 + x_2) \exp\left(-2(x_1^2 + x_2^2)\right) \mathrm{d}\boldsymbol{x}.$$

The following are four IID $\mathcal{N}(0,1)$ random numbers:

$$-0.5379 \quad 1.2537 \quad -1.7361 \quad 0.0204$$

Use these to form, $\hat{\mu}_n$, a Monte Carlo estimate of μ. Granted, n cannot be very large.

Answer: There are multiple ways to approach this problem. One way is to recognize that the integrand looks like a function multipled by a Gaussian PDF with variance $1/4$:

$$\mu = \int_{\mathbb{R}^2} \underbrace{\frac{\pi}{2} \cos(x_1 + x_2)}_{f(\boldsymbol{x})} \underbrace{\frac{\exp\left(-(x_1^2 + x_2^2)/(2 \times \frac{1}{4})\right)}{\sqrt{\left(2\pi \times \frac{1}{4}\right)^2}}}_{\varrho(\boldsymbol{x})} \mathrm{d}\boldsymbol{x}.$$

Here the probability density function is constant, corresponding to the uniform density on the domain.

We must transform our $\mathcal{N}(0,1)$ random numbers to $\mathcal{N}(0,1/4)$ random numbers:

i	1	2	3	4
$Z_i \sim \mathcal{N}(0,1)$	-0.5379	1.2537	-1.7361	0.0204
$X_i = Z_i/2 \sim \mathcal{N}(0,1/4)$	-0.2690	0.6270	-0.8680	0.0102

Next, we order the four X_i to make two vectors in $\mathbb{R}^2$, i.e., $n = 2$. Our Monte Carlo estimate is then the average of the two function values.

i	$\boldsymbol{x}_i$		$f(\boldsymbol{x}_i)$
1	-0.2690	-0.8680	0.6603
2	0.6270	0.0102	1.2625
$\hat{\mu}_2$			0.9614

If we order the four X_i a different way, then we get

i	$\boldsymbol{x}_i$		$f(\boldsymbol{x}_i)$
1	-0.2690	0.6270	1.4712
2	-0.8680	0.0102	1.0275
$\hat{\mu}_2$			1.2493