MATH 565 Monte Carlo Methods in Finance

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Test **Tuesday, November 20**

Instructions:

i. *This test consists of FIVE questions. Answer all of them.*

ii. *The time allowed for this test is 75 minutes*

iii. *This test is closed book, but you may use 1 double-sided letter-size sheets of notes.*

iv. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (25 points)
Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be i.i.d. ordered pairs of random variables where $\mu = E(X_i) = E(Y_i)$, $\sigma^2 = \text{var}(X_i) = \text{var}(Y_i)$, and $\rho = \text{corr}(X_i, Y_i)$ is not necessarily zero. Let $Z = a\bar{X} + b\bar{Y}$, where $\bar{X} = (X_1 + \cdots + X_n)/n$ and $\bar{Y} = (Y_1 + \cdots + Y_n)/n$.

   a) What necessary and sufficient condition on a and b ensures that Z is an *unbiased* estimator of μ?

   Answer: Requiring $\mu = E[Z] = aE[\bar{X}] + bE[\bar{Y}] = (a + b)\mu$ implies that $a + b = 1$ is the necessary and sufficient condition.

   b) What choice of a and b makes Z the unbiased estimate with *smallest variance*? Does this choice of a and b depend on ρ?

   Answer: Since $a + b = 1$, it follows that $b = 1 - a$ and

$$\begin{aligned}\text{var}(Z) &= a^2\,\text{var}(\bar{X}) + b^2\,\text{var}(\bar{Y}) + 2ab\,\text{cov}(\bar{X}, \bar{Y}) = a^2\frac{\sigma^2}{n} + b^2\frac{\sigma^2}{n} + 2ab\frac{\rho\sigma^2}{n} \\ &= \frac{\sigma^2}{n}\left(a^2 + b^2 + 2ab\rho\right) = \frac{\sigma^2}{n}\left(a^2 + (1-a)^2 + 2a(1-a)\rho\right) \\ &= \frac{\sigma^2}{n}\left(2a(1-a)(\rho - 1) + 1\right)\end{aligned}$$

   Since $\rho - 1 \le 0$ we want to maximize $2a(1-a)$, which occurs when $a = b = 1/2$. This choice is independent of ρ. In this case

$$\text{var}(Z) = \frac{\sigma^2}{n}\left(\frac{1}{2}(\rho - 1) + 1\right) = \frac{\sigma^2(\rho + 1)}{2n},$$

   c) What value of ρ makes $\text{var}(Z)$ in the previous part the smallest?

   Answer: When $\rho = -1$, then $\text{var}(Z)$ vanishes.

Note that problems 2–5 are all related.

2. (20 points)
A random variable $Y \sim$ Exponential(λ) is the time in years until the car battery fails. It has the probability density function

$$f(y) = \frac{1}{\lambda} e^{-y/\lambda}, \quad y \geq 0.$$

A *uniform* pseudorandom number generator produces the following output:

$$0.9501, \quad 0.2311, \quad 0.6068, \quad 0.4860, \quad 0.8913.$$

Use these uniform pseudorandom numbers to *produce pseudorandom* Exponential(5) *numbers.*

Answer: We may use the inverse cumulative distribution function method. Note that

$$F(y) = \int_0^y f(t)\,\mathrm{d}t = 1 - e^{-y/5}, \qquad F^{-1}(x) = -5\log(1-x).$$

Letting $x_1, \ldots, x_5$ *denote the uniform pseudorandom numbers above, we get the following* $y_i = -5\log(1-x_i)$*:*

$$14.9916, \quad 1.3142, \quad 4.6677, \quad 3.3275, \quad 11.0958.$$

Using $y_i = -5\log(x_i)$ *is also okay since* $1 - X$ *is a uniform random number if* X *is a uniform random number.*

3. (20 points)
The car battery referred to in the previous problem has a warranty of 5 years. If the time to failure for this car battery, $Y \sim$ Exponential(5), is *less than* 5 *years*, then the manufacturer gives the customer a prorated \$50$(1 - Y/5)$ credit towards the purchase of a new battery (a payoff). What is, μ, the expected amount that the manufacturer needs to credit the customer (the cost of this warranty to the manufacturer)? Assume an interest rate of 0.

Answer:

$$\begin{aligned}
\mu = E[50\max(1 - Y/5, 0)] &= 50\int_0^5 (1 - y/5)\frac{1}{5}e^{-y/5}\,\mathrm{d}y \\
&= 50\left\{ (1 - y/5)(-e^{-y/5})\Big|_0^5 - \int_0^5 \frac{1}{5}e^{-y/5}\,\mathrm{d}y \right\} \\
&= 50\left\{ 1 + \left[e^{-y/5}\right]_0^5 \right\} = 50e^{-1} \approx \$18.
\end{aligned}$$

4. (15 points)
Using the five Exponential(5) pseudorandom numbers generated in Problem 2, find a simple Monte Carlo estimate for μ in Problem 3.

Answer: The Monte Carlo estimate is

$$\begin{aligned}
\hat{\mu} &= \frac{1}{5}\sum_{i=1}^{5} 50\max(1 - y_i/5, 0) \\
&= 10\,(0 + (1 - 1.3142/5) + (1 - 4.6677/5) + (1 - 3.32755) + 0) = \$11.
\end{aligned}$$

5. (20 points)
Construct an importance sampling estimate of μ from Problem 3 by sampling from the distribution Exponential(3) that was defined in Problem 2. Here are five pseudorandom numbers z_i with the distribution Exponential(3):

$$8.9950, \quad 0.7885, \quad 2.8006, \quad 1.9965, \quad 6.6575.$$

Answer: The likelihood ratio is

$$\frac{5^{-1}e^{-z/5}}{3^{-1}e^{-z/3}} = \frac{3e^{2z/15}}{5},$$

and so the importance sampling estimate is

$$\begin{aligned}
\hat{\mu} &= \frac{1}{5}\sum_{i=1}^{5} 50\max(1 - z_i/5, 0)\frac{3e^{2z_i/15}}{5} \\
&= 10\left(0 + (1 - 0.7885/5)\frac{3e^{2\times 0.7885/15}}{5} + (1 - 2.8006/5)\frac{3e^{2\times 2.8006/15}}{5}\right. \\
&\qquad \left. +(1 - 1.9965)\frac{3e^{2\times 1.9965/15}}{5} + 0\right) \\
&\approx \$14.
\end{aligned}$$