MATH 565 Monte Carlo Methods in Finance

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Test **Wednesday, November 12**

Instructions:

i. *This test consists of FOUR questions. Answer all of them.*

ii. *The time allowed for this test is 75 minutes*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *Calculators, even of the programmable variety, are allowed, but computers are not allowed.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (25 marks)
Suppose that $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are independent random variables, where the X_i are identically distributed with mean μ_X and variance σ^2, and the Y_i are identically distributed with mean μ_Y and variance $4\sigma^2$. You estimate $\mu_X + \mu_Y$ by the sum of the sample means, $\bar{X} + \bar{Y}$, where

$$\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n), \qquad \bar{Y} = \frac{1}{m}(Y_1 + \cdots + Y_m).$$

a) Is $\bar{X} + \bar{Y}$ a biased or unbiased estimator of $\mu_X + \mu_Y$?

Answer: $E[\bar{X} + \bar{Y}] = E[\bar{X}] + E[\bar{Y}] = \mu_X + \mu_Y$, *so* $\bar{X} + \bar{Y}$ *is unbiased.*

b) How should m and n be chosen to make $\mathrm{var}(\bar{X} + \bar{Y})$ as small as possible, under the constraint that $m + n = N$ for some fixed budget N? (This is a simple example of importance sampling.)

Answer: Because the X_i *and the* Y_i *are independent,* $\bar{X}$ *and* $\bar{Y}$ *are independent. Thus,*

$$\mathrm{var}(\bar{X} + \bar{Y}) = \mathrm{var}(\bar{X}) + \mathrm{var}(\bar{Y}) = \frac{\sigma^2}{n} + \frac{4\sigma^2}{m} = \sigma^2\left(\frac{1}{n} + \frac{4}{m}\right) = \sigma^2\left(\frac{1}{n} + \frac{4}{N-n}\right).$$

This is smallest when

$$\frac{1}{n} + \frac{4}{N-n}$$

is minimized over all possible n *between* 1 *and* $N-1$. *Taking derivatives with respect to* n *implies*

$$0 = \frac{-1}{n^2} + \frac{4}{(N-n)^2} = -(N-n)^2 + 4n^2,$$

$$2n = N - n, \qquad n = \frac{N}{3}, \qquad m = N - n = \frac{2N}{3}.$$

For the next three problems consider a European call option, i.e., the discounted payoff is

$$\max(S(T) - K, 0)e^{-rT}$$

where $S(t)$ is the stock price as a function of time t in years, T is the time to expiry, K, is the strike price, and r is the risk-free interest rate. The model for the stock price monitored d times takes the form

$$S(j\Delta) = S((j-1)\Delta)e^{(r-\sigma_j^2/2)\Delta+\sigma_j\sqrt{\Delta}X_j}, \quad j = 1, \ldots, d,$$

where $\Delta = T/d$, and $X_1, \ldots, X_d$ are i.i.d. Gaussian (normal) random variables. Specifically we consider the parameter choices:

$$S(0) = K = \$100, \quad T = 0.5, \quad r = 3\%, \quad d = 3.$$

The volatility, σ_j, may be a constant or may depend on S (see below). Consider also the i.i.d. standard Gaussian (normal) pseudorandom numbers:

$$2.1597, \ 0.4500, \ -1.0519, \ \ldots.$$

2. (15 marks)
Compute $S(1/6), S(1/3), S(1/2)$, and the discounted payoff of the European call option for *one* sample path using the model above under the assumption that volatility is *constant*, namely, $\sigma_j = 0.7$.

Answer: Since $rT = 0.015$, $\Delta = 0.5/3 = 1/6$, *and* $(r - \sigma_j^2/2)\Delta = -0.03583$, *and* $\sigma_j\sqrt{\Delta} = 0.2858$, *it follows that*

$$S(j/6) = S((j-1)/6)e^{-0.03583+0.2858X_j}, \quad j = 1, 2, 3.$$

Thus,

$$\begin{aligned}
S(1/6) &= S(0)e^{-0.03583+0.2858X_1} = 100 \times e^{-0.03583+0.2858\times 2.1597} = 178.85, \\
S(1/3) &= S(1/6)e^{-0.03583+0.2858X_2} = 178.85 \times e^{-0.03583+0.2858\times 0.4500} = 196.23, \\
S(1/2) &= S(1/3)e^{-0.03583+0.2858X_3} = 196.23 \times e^{-0.03583+0.2858\times -1.0519} = 140.17, \\
\textit{payoff} &= \max(140.17 - 100, 0)e^{-0.015} = 39.57.
\end{aligned}$$

3. (25 marks)
Compute $S(1/6), S(1/3), S(1/2)$, and the discounted payoff of the European call option for one sample path using the model above under the assumption that the volatility depends on the stock price, namely, the *smile model*:

$$\sigma_j = 0.7 + \left(\frac{S((j-1)\Delta)}{K} - 1\right)^2.$$

Use the same pseudorandom numbers.

Answer: Again, $rT = 0.015$, and $\Delta = 0.5/3 = 1/6$, but now σ_j needs to be updated with along with the stock price according to the formula:

$$\sigma_j = 0.7 + \left(\frac{S((j-1)\Delta)}{100} - 1\right)^2.$$

Thus

$$\begin{aligned}
\sigma_1 &= 0.7 + \left(\frac{S(0)}{100} - 1\right)^2 = 0.7 + \left(\frac{100}{100} - 1\right)^2 = 0.7, \\
S(1/6) &= S(0)e^{(0.03-0.7^2/2)(1/6)+0.7\sqrt{1/6}X_1} = 100 \times e^{-0.03583+0.2858\times 2.1597} = 178.85, \\
\sigma_2 &= 0.7 + \left(\frac{S(1/6)}{100} - 1\right)^2 = 0.7 + \left(\frac{178.85}{100} - 1\right)^2 = 1.32, \\
S(1/3) &= S(1/6)e^{(0.03-1.32^2/2)(1/6)+1.32\sqrt{1/6}X_2} = 198.10, \\
\sigma_3 &= 0.7 + \left(\frac{S(1/3)}{100} - 1\right)^2 = 0.7 + \left(\frac{198.10}{100} - 1\right)^2 = 1.66, \\
S(1/2) &= S(1/3)e^{(0.03-1.66^2/2)(1/6)+1.66\sqrt{1/6}X_3} = 77.45, \\
\textit{payoff} &= \max(77.45 - 100, 0)e^{-0.015} = 0.
\end{aligned}$$

4. (35 marks)
The above European call option for the constant volatility case has the exact price of \$20.15. Suppose that a Monte Carlo simulation for the *constant* volatility case yields discounted payoffs:

$$39.57,\ 86.34,\ 0,\ 28.90,\ 90.43,\ 0,\ 0,\ 0,\ 0,\ 0,$$

and a Monte Carlo simulation utilizing *the same random numbers in the same order* for the European call option under the *smile model* yields discounted payoffs:

$$0,\ 98.02,\ 0,\ 28.73,\ 104.15,\ 0,\ 0,\ 0,\ 0,\ 0.$$

a) What is the estimated European call option price assuming *constant* volatility?

Answer:

$$\bar{x} = \frac{1}{10}(39.57 + 86.34 + 28.90 + 90.43) = \$24.54$$

b) What is the estimated European call option price assuming a volatility *smile*?

Answer:

$$\bar{y} = \frac{1}{10}(98.02 + 28.73 + 104.15) = \$23.10$$

c) Construct a *control variate* approximation to the European call option price assuming a volatility smile. Give a numerical answer.

Answer: The control variate approximation is

$$\bar{y} + \beta(\mu_X - \bar{x}) = 23.10 + \beta(20.15 - 24.54),$$

where the best estimate for β *is*

$$\begin{aligned}
\hat{\beta} &= \frac{\sum_{i=1}^{10}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{10}(x_i - \bar{x})^2} \\
&= [(39.57 - 24.54)(0 - 23.10) + (86.34 - 24.54)(98.02 - 23.10) \\
&\qquad +(28.90 - 24.54)(28.73 - 23.10) + (90.43 - 24.54)(104.15 - 23.10) \\
&\qquad +6(0 - 24.54)(0 - 23.10)] \Big/ \big[(39.57 - 24.54)^2 + (86.34 - 24.54)^2 \\
&\qquad +(28.90 - 24.54)^2 + (90.43 - 24.54)^2 + 6(0 - 24.54)^2\big] \\
&= 1.086.
\end{aligned}$$

Thus, the control variate estimate is $23.10 + 1.086(20.15 - 24.54) = \$18.35$.