# MATH 565 Monte Carlo Methods in Finance

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**Test** **Monday, November 9**

*Instructions:*

i. *This test consists of* FOUR *questions. You must answer the* FIRST TWO PLUS EITHER ONE OF THE LAST TWO. *If you attempt both of the last two questions, the larger of your two scores will be counted.*

ii. *The time allowed for this test is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *Calculators, even of the programmable variety, are allowed. Computers, but only using MATLAB or JMP, are also allowed. No internet access.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (30 points)
Consider the problem of pricing a European put option. The stock price is modeled by a geometric Brownian motion and is monitored quarterly for one year:

$$S(0) = 100, \qquad S(0.25(j+1)) = S(0.25j)e^{(r-\sigma^2/2)0.25+0.5\sigma X_j}, \quad j = 0, 1, 2, 3,$$

where $r = 3\%$ and $\sigma = 50\%$. Consider also the following string of normal (Gaussian) pseudorandom numbers:

$$0.53767, -1.34989, 0.67150, 0.88840, \ldots$$

a) Using the string of pseudorandom numbers above, compute one stock path:

$$S(0), S(1/4), S(1/2), S(3/4), S(1).$$

*Answer: Since* $(r - \sigma^2/2)0.25 = (0.03 - 0.125)0.25 = -0.02375$ *and* $0.5\sigma = 0.25$, *it follows that*

$$\begin{aligned} S(0.25(j+1)) &= S(0.25j)e^{-0.02375+0.25X_j}, \quad j = 0, 1, 2, 3, \\ S(0.25) &= S(0)e^{-0.02375+0.25X_j} = 111, \\ S(0.5) &= S(0.25)e^{-0.02375+0.25X_j} = 77, \\ S(0.75) &= S(0.5)e^{-0.02375+0.25X_j} = 89, \\ S(1) &= S(0.75)e^{-0.02375+0.25X_j} = 108. \end{aligned}$$

b) Use the stock path that you have created above plus the other nine paths below to approximate the price of a European put option with a strike price of \$90 using simple Monte

Carlo.

| $S(0)$ | $S(0.25)$ | $S(0.5)$ | $S(0.75)$ | $S(1)$ |
|---|---|---|---|---|
| | | insert your path here | | |
| 100 | 154 | 320 | 230 | 168 |
| 100 | 55 | 65 | 75 | 56 |
| 100 | 121 | 116 | 169 | 134 |
| 100 | 105 | 123 | 135 | 63 |
| 100 | 70 | 65 | 82 | 114 |
| 100 | 87 | 82 | 96 | 101 |
| 100 | 106 | 150 | 135 | 109 |
| 100 | 238 | 329 | 345 | 473 |
| 100 | 194 | 270 | 215 | 137 |

*Answer: Computing the discounted payoffs for these ten paths we get:*

| $S(0)$ | $S(0.25)$ | $S(0.5)$ | $S(0.75)$ | $S(1)$ | $\max(90 - S(1), 0)e^{-r}$ |
|---|---|---|---|---|---|
| 100 | 111 | 77 | 89 | 108 | 0 |
| 100 | 154 | 320 | 230 | 168 | 0 |
| 100 | 55 | 65 | 75 | 56 | 33 |
| 100 | 121 | 116 | 169 | 134 | 0 |
| 100 | 105 | 123 | 135 | 63 | 26 |
| 100 | 70 | 65 | 82 | 114 | 0 |
| 100 | 87 | 82 | 96 | 101 | 0 |
| 100 | 106 | 150 | 135 | 109 | 0 |
| 100 | 238 | 329 | 345 | 473 | 0 |
| 100 | 194 | 270 | 215 | 137 | 0 |

*The average of these payoffs is* $(33 + 26)/10 = \$5.9$*, which is the estimated European option price.*

2. (30 points)
   Now consider two other options.

   a) The first is a down and in put option with a strike price of \$90 and a barrier of \$80. The option pays off only if the stock price becomes lower than the barrier at some time. Estimate the price of the barrier put option using the ten stock paths in the previous problem. Is the estimated price higher, lower or the same as the estimated price of the European put option? Do you expect the true price of the barrier put option to be higher, lower or the same as the true price of the European put option? Why? If the Monte Carlo estimation does not align with what happens for the true prices, explain why.

      *Answer: All the paths that pay off for the European option also pay off for the barrier option. Thus, their estimated prices are the same. However, we expect the true price of the barrier option to be lower than that of the European option because there are more conditions for a positive payoff. In this case we only looked at ten sample paths, a small sample, so the Monte Carlo estimates did not align with the true prices.*

b) The second is an American put option with a strike price of \$90 and an exercise boundary of

| $t$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|---|
| boundary | 55 | 60 | 66 | 75 | ?? |

What is the value of the exercise boundary for $t = 1$? Estimate the price of the American put option using the ten stock paths in the previous problem. Is the estimated price higher, lower or the same as the estimated price of the European put option? Do you expect the true price of the American put option to be higher, lower or the same as the true price of the European put option? Why? If the Monte Carlo estimation does not align with what happens for the true prices, explain why.

*Answer: The exercise boundary for t* $= 1$ *is* 90*, the strike price. Let* $\tau$ *be the exercise time for each path. Then*

| $S(0)$ | $S(0.25)$ | $S(0.5)$ | $S(0.75)$ | $S(1)$ | $\tau$ | $\max(90 - S(\tau), 0)e^{-r\tau}$ |
|---|---|---|---|---|---|---|
| 100 | 111 | 77 | 89 | 108 | 1.00 | 0 |
| 100 | 154 | 320 | 230 | 168 | 1.00 | 0 |
| 100 | 55 | 65 | 75 | 56 | 0.25 | 34 |
| 100 | 121 | 116 | 169 | 134 | 1.00 | 0 |
| 100 | 105 | 123 | 135 | 63 | 1.00 | 26 |
| 100 | 70 | 65 | 82 | 114 | 0.50 | 25 |
| 100 | 87 | 82 | 96 | 101 | 1.00 | 0 |
| 100 | 106 | 150 | 135 | 109 | 1.00 | 0 |
| 100 | 238 | 329 | 345 | 473 | 1.00 | 0 |
| 100 | 194 | 270 | 215 | 137 | 1.00 | 0 |

*The average of these payoffs is* $(34+26+25)/10 = \$8.5$*, which is the estimated American option price. This is higher than the estimated European price. This is in line with what one expects of the true prices. Since the American option may be exercised at any time, its price is greater than that of the European option.*

3. (40 points)
Let $X$ be a Gamma($a$) random variable, which means that its probability density function and moment generating functions are

$$f(x) = \frac{1}{\Gamma(a)} x^a e^{-x}, \; x \geq 0, \qquad M(t) = E[e^{tX}] = \frac{1}{(1-t)^a}, \; t < 1,$$

where $\Gamma(a) = \int_0^\infty x^a e^{-x}\,\mathrm{d}x$. Let $X_1, X_2$ be i.i.d. Gamma($a$) for some fixed $a \geq 0$, and let $S(1) = S(0)e^{c+b(X_1 - X_2)}$ be your model for the stock price one year later for some fixed $b$. What should be the value of $c$ in terms of $a$, $b$, and the interest rate $r$ to ensure no arbitrage opportunities?

*Answer: To ensure no arbitrage opportunities we must have*

$$\begin{aligned}
S(0)e^r &= E[S(1)] = E[S(0)e^{c+b(X_1-X_2)}] = S(0)e^c E[e^{b(X_1-X_2)}] \\
&= S(0)e^c E[e^{bX_1}]E[e^{-bX_2}] \quad \textit{since } X_1,\ X_2 \textit{ are independent,} \\
&= S(0)e^c M(b)M(-b) = S(0)e^c \frac{1}{(1-b)^a}\frac{1}{(1+b)^a} = \frac{S(0)e^c}{(1-b^2)^a} \\
e^{c-r} &= (1-b^2)^a \\
c &= r + a\log(1-b^2)
\end{aligned}$$

4. (40 points)
Consider the multivariate integration problem

$$\mu = \int_0^1 \int_0^1 \exp(x_1 x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 = ?$$

Approximate this integral with an absolute error of 0.01 or less by using: i) simple Monte Carlo simulation, and ii) Monte Carlo simulation with the control variates $x_1$ and $x_2$ together. For your chosen sample size, what is the ratio of simple Monte Carlo error to the error obtained using control variates?

*Answer: Below is the code to estimate $\mu$ by simple Monte Carlo and control variates. You need a sample of about $n = 4000$ to get the error $< 0.01$ for simple Monte Carlo, and a sample of about $n = 800$ to get the error $< 0.01$ for control variates. For $n = 4000$ the error of simple Monte Carlo is about 2.3 times that of control variates.*

```
%% Control variates
n=1e4; %sample size
f=@(x) exp(x(:,1).*x(:,2)); %define function to be integrated
x=rand(n,2); %uniform random samples
mux=0.5*ones(1,2); %true means of x_1 and x_2
xbar=mean(x,1); %sample averages of x_1 and x_2
G=x-repmat(xbar,n,1); %x - xbar in matrix form
y=f(x); %function values
ybar=mean(y) %sample mean of function values
ciMCwidth=1.96*std(y)/sqrt(n) %confidence interval for simple MC
beta=G\(y-ybar); %beta vector for control variates
muhat=ybar+(mux-xbar)*beta %control variate estimator
resid=y-ybar-G*beta; %residuals from regression
stddev=std(resid); %standard error of residuals
ciCVwidth=1.96*stddev/sqrt(n) %confidence interval width for control variates
errratio=ciMCwidth/ciCVwidth %ratio of errors
```

```
n =

        4000
```

```
ybar =
   1.310434395333645
ciMCwidth =
   0.009849700594992
muhat =
   1.314074388245360
ciCVwidth =
   0.004377475231410
errratio =
   2.250087110560029
```