MATH 565 Monte Carlo Methods in Finance

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Test **Tuesday, November 9**

Instructions:

i. This test consists of THREE questions. Answer all of them.

ii. The time allowed for this test is 75 minutes

iii. This test is closed book, but you may use 4 double-sided letter-size sheets of notes.

iv. Calculators, even of the programmable variety, are allowed. Computers, but only using MATLAB or JMP, are also allowed. No internet access.

v. Show all your work to justify your answers. Answers without adequate justification will not receive credit.

1. (30 marks)
Let $X_1, X_2, \ldots$ be independent and identically distributed random variables on $\mathbb{R}$ with probability density f_X. Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables on $\mathbb{R}$ with probability density f_Y. Let $g : \mathbb{R} \to \mathbb{R}$ be some function, and suppose that you wish to estimate

$$I = \int_{\mathbb{R}} g(x)\, dx.$$

a) Which one or more of the following are *unbiased estimators* of I:

$$\hat{I}_1 = \frac{1}{n}\sum_{i=1}^{n} g(X_i), \qquad \hat{I}_2 = \frac{1}{n}\sum_{i=1}^{n} \frac{g(X_i)}{f_X(X_i)}, \qquad \hat{I}_3 = \frac{1}{n}\sum_{i=1}^{n} g(X_i) f_X(X_i),$$
$$\hat{I}_4 = \frac{1}{n}\sum_{i=1}^{n} g(Y_i), \qquad \hat{I}_5 = \frac{1}{n}\sum_{i=1}^{n} \frac{g(Y_i)}{f_Y(Y_i)}, \qquad \hat{I}_6 = \frac{1}{n}\sum_{i=1}^{n} g(Y_i) f_Y(Y_i)?$$

Answer: Only I_2 *and* I_5, *because*

$$E[\hat{I}_1] = \frac{1}{n}\sum_{i=1}^{n} E[g(X_i)] = E[g(X)] = \int_{\mathbb{R}} g(x)\, f_X(x)\, \mathrm{d}x \neq I$$
$$E[\hat{I}_2] = \frac{1}{n}\sum_{i=1}^{n} E\left[\frac{g(X_i)}{f_X(X_i)}\right] = E\left[\frac{g(X)}{f_X(X)}\right] = \int_{\mathbb{R}} \frac{g(x)}{f_X(x)}\, f_X(x)\, \mathrm{d}x = I,$$
$$E[\hat{I}_3] = \frac{1}{n}\sum_{i=1}^{n} E[g(X_i) f_X(X_i)] = \int_{\mathbb{R}} g(x) f_X(x)\, f_X(x)\, \mathrm{d}x \neq I,$$
$$E[\hat{I}_4] = \frac{1}{n}\sum_{i=1}^{n} E[g(Y_i)] = E[g(Y)] = \int_{\mathbb{R}} g(y)\, f_Y(y)\, \mathrm{d}y \neq I,$$
$$E[\hat{I}_5] = \frac{1}{n}\sum_{i=1}^{n} E\left[\frac{g(Y_i)}{f_Y(Y_i)}\right] = E\left[\frac{g(Y)}{f_Y(Y)}\right] = \int_{\mathbb{R}} \frac{g(y)}{f_Y(y)}\, f_Y(y)\, \mathrm{d}y = I,$$
$$E[\hat{I}_6] = \frac{1}{n}\sum_{i=1}^{n} E[g(Y_i) f_Y(Y_i)] = E\left[g(Y) f_Y(Y)\right] = \int_{\mathbb{R}} g(y) f_Y(y)\, f_Y(y)\, \mathrm{d}y \neq I.$$

b) Express the variance(s) of the *unbiased estimator(s)* from the previous part in terms of integrals involving g, f_X, and f_Y. If you have some knowledge of the values of these integrals, which estimator would you choose?

Answer:

$$\begin{aligned}
\operatorname{var}(\hat{I}_2) &= \operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{g(X_i)}{f_X(X_i)}\right) = \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{var}\left(\frac{g(X_i)}{f_X(X_i)}\right) = \frac{1}{n}\operatorname{var}\left(\frac{g(X)}{f_X(X)}\right) \\
&= \frac{1}{n}\int_{\mathbb{R}}\left[\frac{g(x)}{f_X(x)} - I\right]^2 f_X(x)\,\mathrm{d}x = \frac{1}{n}\int_{\mathbb{R}}\left[\frac{g(x)}{f_X(x)} - \int_{\mathbb{R}} g(t)\,\mathrm{d}t\right]^2 f_X(x)\,\mathrm{d}x, \\
\operatorname{var}(\hat{I}_5) &= \operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{g(Y_i)}{f_Y(Y_i)}\right) = \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{var}\left(\frac{g(Y_i)}{f_Y(Y_i)}\right) = \frac{1}{n}\operatorname{var}\left(\frac{g(Y)}{f_Y(Y)}\right) \\
&= \frac{1}{n}\int_{\mathbb{R}}\left[\frac{g(y)}{f_Y(y)} - I\right]^2 f_Y(y)\,\mathrm{d}y = \frac{1}{n}\int_{\mathbb{R}}\left[\frac{g(y)}{f_Y(y)} - \int_{\mathbb{R}} g(t)\,\mathrm{d}t\right]^2 f_Y(y)\,\mathrm{d}y.
\end{aligned}$$

The better estimator would be the one with the smaller variance.

2. (35 marks)
Consider a model for the asset price over the course of one year, where $r = 0$, $S(0) = 100$, and the volatility, σ, depends on the asset price. Specifically,

$$S(t_j) = S(t_{j-1})\exp\left(-\frac{\sigma^2(S(t_{j-1}))}{2d} + \sigma(S(t_{j-1}))\sqrt{\frac{1}{d}}\,X_j\right), \quad j = 1,\dots,d,$$
$$X_1, X_2, \dots, X_d \text{ i.i.d. } N(0,1),$$
$$t_j = j/d, \quad j = 0,\dots,d,$$
$$\sigma(S) = 0.5 + 2\text{E}{-5}(S-100)^2.$$

Use i.i.d. sampling with $n = 10^5$ samples and $d = 52$ to approximate the price of a European call option with strike price \$100. Also estimate the error of your approximation.

Answer: The MATLAB program that solves this problem is

```
r=0; %interest rate
sig= @(S) 0.5 + 2e-5*(S-100).^2; %volatility function
d=52; %number of discretizations
S0=100; %initial asset price
K=100; %strike price
n=1e5; %number of samples
T=1; %time to expiry
Delta=T/d; %time step
Smat=[repmat(S0,n,1) zeros(n,d)]; %initialize stock paths
Xmat=randn(n,d); %get normal random numbers
for j=1:d
    sigj=sig(Smat(:,j)); %compute variance depending on asset price
    Smat(:,j+1)=Smat(:,j).*exp((r-sigj.^2/2)*Delta + sigj.*Xmat(:,j)*sqrt(Delta)); %next s
```

```
end
payoff=max(Smat(:,d+1)-K,0)*exp(-r*T); %payoff
call=mean(payoff); %approximate call price
err=1.96*std(payoff)/sqrt(n); %estimate of error
disp('The price of the call option with varying volatility')
disp(['     is $' num2str(call) ' +/- ' num2str(err)])

The price of the call option with varying volatility
     is $17.3922 +/- 0.25137
```

3. (35 marks)
Consider the computation of a multivariate normal probability

$$p = \int_{[-1,1]^3} \frac{1}{\sqrt{(2\pi)^3 \det(\mathsf{C})}} \exp\left(-\frac{1}{2}\boldsymbol{x}^T \mathsf{C}^{-1} \boldsymbol{x}\right) \mathrm{d}\boldsymbol{x}, \quad \text{where } \mathsf{C} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

Estimate the value of p by Monte Carlo using $n = 10^2, \ldots, 10^5$ samples of i) i.i.d. sampling, and ii) either antithetic variates or Latin hypercube sampling. (Hint: Try your program on the problem $\int_{[-1,1]^3} 1 \, \mathrm{d}\boldsymbol{x}$ as a check.)

Answer: The MATLAB program for this problem is

```
%% Problem 3 on Test 2
d=3;
C=[2 -1 0; -1 3 1; 0 1 4];
eig(C) %to check that it is positive definite
integrand=@(x,C) exp(-sum(x.*(C\x')',2)/2)/sqrt((2*pi)^3*det(C));
nvec=10.^(2:5)';
nn=size(nvec,1);
for i=1:nn
    n=nvec(i); %sample size
    disp(['For a sample size of n = ' int2str(n) ' the estimate of p'])
    nov2=n/2; %half the sample size
    xiid=2*rand(n,d)-1; %iid sampling
    yiid=integrand(xiid,C); %integrand values
    piid=8*mean(yiid); %value of the integral
    disp(['   is ' num2str(piid) ' using iid sampling'])
    temp=rand(nov2,d);
    xanti=[2*temp-1; 1-2*temp]; %antithetic variate sampling
    yanti=integrand(xanti,C); %integrand values
    panti=8*mean(yanti); %value of the integral
    disp(['   is ' num2str(panti) ' using antithetic variates'])
    xlhs=2*lhsdesign(n,d,'criterion','none')-1; %lhs sampling
    ylhs=integrand(xlhs,C); %integrand values
    plhs=8*mean(ylhs); %value of the integral
    disp(['   is ' num2str(plhs) ' using Latin hypercube sampling'])
end
```

```
ans =
   1.267949192431123
   3.000000000000001
   4.732050807568877
For a sample size of n = 100 the estimate of p
   is 0.094691 using iid sampling
   is 0.09544 using antithetic variates
   is 0.097473 using Latin hypercube sampling
For a sample size of n = 1000 the estimate of p
   is 0.096702 using iid sampling
   is 0.096738 using antithetic variates
   is 0.096672 using Latin hypercube sampling
For a sample size of n = 10000 the estimate of p
   is 0.096815 using iid sampling
   is 0.096936 using antithetic variates
   is 0.096858 using Latin hypercube sampling
For a sample size of n = 100000 the estimate of p
   is 0.09696 using iid sampling
   is 0.096961 using antithetic variates
   is 0.096879 using Latin hypercube sampling
For a sample size of n = 1000000 the estimate of p
   is 0.096831 using iid sampling
   is 0.096867 using antithetic variates
   is 0.09685 using Latin hypercube sampling
```