MATH 565 Monte Carlo Methods in Finance

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Test **Tuesday, November 8**

Instructions:

i. *This test consists of THREE questions. Answer all of them.*

ii. *The time allowed for this test is 75 minutes*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *Calculators, even of the programmable variety, are allowed. Computers, but only using MATLAB or JMP, are also allowed. No internet access.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (45 points)
Consider a stock price modeled as a geometric Brownian motion,

$$\mathrm{d}\log(S) = (r - \sigma^2/2)\mathrm{d}t + \sigma \mathrm{d}B(t), \qquad S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma B(t)},$$

where t is time in years, $S(0) = 70$ is the initial stock price, $r = 1\%$ is the interest rate, $\sigma = 50\%$ is the volatility, and $B(\cdot)$ is a Brownian motion. We want to price an American put option that expires in $T = 1$ year and has a strike price of $K = 100$. (Actually, this is a Bermudan option because we only look at quarterly time intervals, $\Delta = 1/4$.)

a) Use the *normal (Gaussian)* pseudorandom numbers

$$0.3018, \quad 0.3999, \quad -0.9300, \quad -0.1768$$

to simulate *one* stock path, $S(t_j)$, $j = 0, \ldots, 4$ at the times, $t_j = 0, 1/4, 1/2, 3/4$ and 1.

Answer: For the Brownian motion, with $t_j = j/4$

$$B(j/4) = B((j-1)/4) + \sqrt{1/4}X_j = B((j-1)/4) + (1/2)X_j, \quad j = 1, \ldots, 4$$

j	0	1	2	3	4
$t_j = j/4$	0	1/4	1/2	3/4	1
X_j		0.3018	0.3999	−0.9300	−0.1768
$B(t_j)$	0	0.1509	0.3509	−0.1141	−0.2025
$S(t_j)$	70.0	73.3	78.8	60.7	56.4

b) Explain why knowing *one* stock path is insufficient information to determine the discounted payoff for that one path.

Answer: An American/Bermudan option has the option of early exercise. The exercise boundary is needed to determine when to exercise, and the estimate of this exercise boundary requires many paths.

c) The table below gives the values of the *exercise boundary*, $b(t)$.

t	0	1/4	1/2	3/4	1
$b(t)$	41	46	50	62	??

What is the meaning of $b(t)$? What is the value of $b(1)$? What is the discounted payoff of the put option for the stock path you generated above?

Answer: If at any time, t, it happens that $S(t) < b(t)$, then the holder of the put option should exercise it, because the expected payoff from holding the option is less than that obtained by exercising immediately. On the other hand, if $S(t) > b(t)$ then it is more profitable, on average, to continue to hold the put option.

At $t = 1$, the time of expiry, the exercise boundary is 100*, the strike price.*

According to the table, the stock path first falls below the exercise boundary at $t = 3/4$. The discounted payoff is

$$(100 - S(3/4))e^{-0.01\times(3/4)} = 39.3e^{-0.0075} = 39.1$$

d) Now suppose that $S(0) = 50$ instead of $S(0) = 70$, but everything else is the same. How does this affect the stock price path, the exercise boundary, the time to exercise, and the discounted payoff, if at all?

Answer: Because only the initial stock price changes, the new stock path is just the old one times $50/70 = 5/7$

j	0	1	2	3	4
$t_j = j/4$	0	1/4	1/2	3/4	1
X_j		0.3018	0.3999	−0.9300	−0.1768
$B(t_j)$	0	0.1509	0.3509	−0.1141	−0.2025
$S(t_j)$	50.0	52.4	56.3	43.3	40.3

The exercise boundary is unchanged, and the stock path crosses at the same time (in this case). The discounted payoff is now 56.3*.*

2. (15 points)
Consider the previous problem with $S(0) = 70$, and with S satisfying the above stochastic differential equation, but now suppose that the volatility, σ, depends on the asset price as follows:

$$\sigma(S) = 0.5 + (S/100 - 1)^2$$

The stochastic differential equation no longer has an analytic solution. Recompute the one stock price path with the same time step, but using the Euler approximation.

Answer: The Euler approximation with $t_j = j/4$ gives

$$\log(S(t_j)) = \log(S(t_{j-1})) + (r - \sigma^2(S(t_{j-1}))/2)(1/4) + \sigma(S(t_{j-1}))\sqrt{1/4}X_j$$

j	0	1	2	3	4
$t_j = j/4$	0	1/4	1/2	3/4	1
X_j		0.3018	0.3999	−0.9300	−0.1768
$\sigma(S(t_j))$	0.590	0.571	0.543	0.664	
$S(t_j)$	70.0	73.4	79.2	59.5	53.2

3. (40 points)
Consider the integral

$$\int_{-1}^{1}\int_{-1}^{1} \cos(xy)\,\mathrm{d}x\mathrm{d}y.$$

a) Use a simple Monte Carlo method to compute the value of this integral with an absolute error of 0.001.

b) Since $\cos(xy) \approx 1 - (xy)^2/2$, this suggests that x^2y^2 may be a good control variate for this integral. Use this control variate for the same number of samples that you used in part a) to get a more accurate answer for the integral. What is the error estimate of your answer using control variates?

Answer: Note that this integral may be written as

$$\mu = \int_{-1}^{1}\int_{-1}^{1} 4\cos(xy)\,\frac{1}{4}\,\mathrm{d}x\mathrm{d}y,$$

where $1/4$ is the probability density function of the uniform distribution on $[-1,1]^2$. So we take $f(x) = 4\cos(xy)$ to be our integrand. By trial and error, $n = 5 \times 10^5$ samples is sufficient. The control variate $Z = X^2Y^2$ has a mean of $1/9$. Note that control variates makes the error smaller by a factor of 50.

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%% Problem 3

%Simple Monte Carlo
n=5e5; %sample size
x=2*rand(n,2)-1; %uniform random numbers on [-1,1]
fx=4*cos(x(:,1).*x(:,2)); %integrand values
appxinteg=mean(fx);
err=1.96*std(fx)/sqrt(n);
disp(['The integral by simple Monte Carlo is ' num2str(appxinteg,'%3.6f') ...
    ' +/- ' num2str(err,'%3.6f')])

%Using control variates
cv=(x(:,1).*x(:,2)).^2; %control variate samples
meancv=mean(cv); %sample mean of the control variate
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cvintex=1/9; %true mean of the control variate
tempa=cv-meancv; %control variate samples minus sample mean
beta=tempa\(fx-appxinteg)%use regression to find the best beta
appxintegcv=appxinteg-(meancv-cvintex)*beta; %control variate estimate of the integral
tempb=fx-appxinteg-tempa*beta; %residuals
errcv=1.96*sqrt(sum(tempb.*tempb)/((n-1)*n)); %error estimate for control variates
disp(['The integral by control variates is ' num2str(appxintegcv,'%3.6f')...
    ' +/- ' num2str(errcv,'%3.6f')])

The integral by simple Monte Carlo is 3.784196 +/- 0.000881
The integral by control variates is 3.784351 +/- 0.000017
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