MATH 565 Monte Carlo Methods in Finance

Fred J. Hickernell **Test 2** **Wednesday, October 30**

Instructions:

i. *This test consists of FOUR questions. Answer as many as you would like. Your score will be based on your BEST THREE answers.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text. No internet access.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (33 marks)
Consider the problem of pricing an American put option with a strike price of \$50 that expires in 6 months (26 weeks). The option is monitored weekly. The stock price follows a geometric Brownian motion with a zero interest rate. The exercise boundary $b(t)$ (dollars), and some stock paths, $S_i(t)$, are given in the table below:

t (weeks)	18	19	20	21	22	23	24	25	26
$b(t)$	33.57	33.83	34.21	34.51	34.93	35.61	36.25	37.97	50.00
$S_1(t)$	38.45	37.12	37.94	36.46	35.90	36.53	39.69	43.68	43.45
$S_2(t)$	35.92	38.12	34.17	34.05	30.40	29.05	27.11	28.53	29.83
$S_3(t)$	53.94	59.59	62.42	64.02	74.03	77.26	77.67	77.88	75.56
$S_4(t)$	38.28	32.00	34.57	35.17	38.15	39.52	38.71	39.76	37.80
$S_5(t)$	38.96	38.71	41.15	43.12	41.45	42.86	43.21	44.79	50.29
$S_6(t)$	37.80	33.14	27.65	26.65	24.80	24.43	27.23	26.88	25.00

Assume that none of these paths is exercised before week 18.

a) Use an optimal exercise strategy (the one used to price American options) to determine whether or not each path is exercised, when it is exercised, and the payoff.

Answer: The paths should be exercised at the first time that their price falls below the exercise boundary. This corresponds to

t	18	19	20	21	22	23	24	25	26	
$b(t)$	33.57	33.83	34.21	34.51	34.93	35.61	36.25	37.97	50.00	*payoff*
$S_1(t)$									43.45	6.55
$S_2(t)$			34.17							15.83
$S_3(t)$					**never**					0
$S_4(t)$		32.00								18.00
$S_5(t)$					**never**					0
$S_6(t)$		33.14								16.86

b) What is the Monte Carlo estimate of the American put option price based only on these six paths?

Answer:

$$\frac{1}{6}(6.55 + 15.83 + 0 + 18.00 + 0 + 16.86) = 9.54.$$

c) Which of these paths is exercised at the time corresponding to its largest possible payoff assuming perfect knowledge of the future? Which paths with a zero payoff could have been exercised for a positive payoff if the option holder had not waited?

Answer: Only path 4 obtained the largest possible payoff assuming perfect future knowledge. Path 5 could have been exercised for a positive payoff, but the expected value of holding at every step was always more than the payoff.

2. (33 marks)
An asset price follows model with a skew volatility and an interest rate of 1%:

$$\mathrm{d}S(t) = 0.01S(t)\mathrm{d}t + \sigma(S(t))S(t)\mathrm{d}B(t), \qquad \sigma(S) = 0.5 + 0.2\left(\frac{S}{50} - 1\right), \qquad S(0) = \$50.$$

In the above equation the unit of time one year. Consider a one year time horizon and quarterly time stepping. Using the single instance of the Brownian motion below construct a single path for S:

t (in years)	0	0.25	0.5	0.75	1
$B(t)$	0.0000	−0.8984	0.3112	0.8198	0.6236

Answer: We use the Euler time stepping for $\log(S(t))$, *which implies that*

$$t_j = 0.25j, \quad j = 0, 1, 2, 3, 4$$

$$\sigma_j = 0.5 + 0.2\left(\frac{S(t_j)}{50} - 1\right)$$

$$S(t_{j+1}) = S(t_j)\exp((0.01 - \sigma_j^2/2)/4 + \sigma_j(B(t_{j+1}) - B(t_j))$$

j	0	1	2	3	4
t_j *(in years)*	0	0.25	0.5	0.75	1
$B(t_j)$	0.0000	−0.8984	0.3112	0.8198	0.6236
$B(t_{j+1}) - B(t_j)$		−0.8984	1.2096	0.5086	−0.1962
$S(t_j)$	50.00	31.00	50.75	63.67	55.09
σ_j	0.5000	0.4240	0.5030	0.5547	

3. (33 marks)
Consider the problem of computing $\mu = \mathbb{E}(e^X)$, where $X \sim \mathcal{N}(0, 1)$. If you use importance sampling with $Z \sim \mathcal{N}(a, 1)$ for some a, what value of a gives the smallest variance of the importance sampling estimator?

Answer: Note that

$$\begin{aligned}
\mu &= \mathbb{E}[\mathrm{e}^X] \\
&= \int_{-\infty}^{\infty} \mathrm{e}^x \times \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \mathrm{e}^x \frac{\mathrm{e}^{-x^2/2}}{\mathrm{e}^{-(x-a)^2/2}} \times \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(x-a)^2/2}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{(1-a)x+a^2/2} \times \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(x-a)^2/2}\,\mathrm{d}x \\
&= \mathbb{E}[\mathrm{e}^{(1-a)Z+a^2/2}]
\end{aligned}$$

The variance of $\mathrm{e}^{(1-a)Z+a^2/2}$ *is smallest when* $a = 1$*, in which case* $\mathrm{e}^{(1-a)Z+a^2/2}$ *is a constant with zero variance.*

4. (33 marks)
Consider the problem of computing $\mu = \mathbb{E}(\mathrm{e}^X)$, where $X \sim \mathcal{U}[0, 1]$. What are the variances of the simple Monte Carlo and the antithetic estimators using n function values? Which variance is smaller?

Answer: First, we consider $Y = \mathrm{e}^X$*. In this case,*

$$\begin{aligned}
\mathbb{E}(Y) &= \int_0^1 \mathrm{e}^x\,\mathrm{d}x = \mathrm{e} - 1, \\
\mathbb{E}(Y^2) &= \int_0^1 \mathrm{e}^{2x}\,\mathrm{d}x = \frac{1}{2}(\mathrm{e}^2 - 1), \\
\mathrm{var}(Y) &= \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \frac{1}{2}(\mathrm{e}^2 - 1) - (\mathrm{e} - 1)^2 = \frac{-\mathrm{e}^2 + 4\mathrm{e} - 3}{2} = \frac{(3 - \mathrm{e})(\mathrm{e} - 1)}{2} \\
&= 0.2420
\end{aligned}$$

The variance of the simple Monte Carlo estimator using n *samples is* $0.2420/n$*.*
Next, we consider $\widetilde{Y} = (\mathrm{e}^X + \mathrm{e}^{1-X})/2$*. In this case,*

$$\begin{aligned}
\mathbb{E}(\widetilde{Y}) &= \mathbb{E}(Y) = \mathrm{e} - 1, \\
\mathbb{E}(\widetilde{Y}^2) &= \int_0^1 \frac{1}{4}(\mathrm{e}^x + \mathrm{e}^{1-x})^2\,\mathrm{d}x = \frac{1}{4}\int_0^1 (\mathrm{e}^{2x} + 2\mathrm{e} + \mathrm{e}^{2-2x})\,\mathrm{d}x \\
&= \frac{1}{4}\left[\frac{1}{2}(\mathrm{e}^2 - 1) + 2\mathrm{e} + \frac{1}{2}(\mathrm{e}^2 - 1)\right] = \frac{1}{4}(\mathrm{e}^2 + 2\mathrm{e} - 1), \\
\mathrm{var}(\widetilde{Y}) &= \mathbb{E}(\widetilde{Y}^2) - \mathbb{E}(\widetilde{Y})^2 = \frac{1}{4}(\mathrm{e}^2 + 2\mathrm{e} - 1) - (\mathrm{e} - 1)^2 = \frac{-3\mathrm{e}^2 + 10\mathrm{e} - 5}{4} = 0.0039
\end{aligned}$$

The variance of the antithetic variates estimator using $n/2$ *samples of* $\widetilde{Y}$*, or* n *samples of* e^x *is* $0.0039/(n/2) = 0.0078/n$*. The antithetic variates estimator is more efficient.*