MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test consists of FOUR questions worth a total of 100 marks. Answer all of them.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit. Write out pseudo-code for the programs that you run to get your answers.*

vi. *Off-site students may contact the instructor at 630-696-8124.*

1. (15 points)
Consider the following IID $\mathcal{N}(0,1)$ random variables X_j that come from a pseudo-random number generator:

j	1	2	3	4
X_j	-1.3011	0.9032	-0.0339	1.3695

Use these random numbers to construct the values of a Brownian motion, B_1, at weekly times for four weeks. Specifically, find $B_1(t_j)$ for $t_j = j/52$ years with $j = 0, 1, 2, 3, 4$.

Answer: We use the iterative process

$$B(0) = 0, \qquad B(t_j) = B(t_{j-1}) + \sqrt{t_j - t_{j-1}} X_j = B(t_{j-1}) + \sqrt{1/52} X_j, \qquad j = 1, \ldots, 4$$

to obtain

j	0	1	2	3	4
X_j		-1.3011	0.9032	-0.0339	1.3695
$B_1(t_j)$	0	-0.1804	-0.0552	-0.0599	0.1300

2. (40 points)
Consider a stock that is modeled by a geometric mean Brownian motion with initial price $S(0) = \$45$, interest rate 1%, and volatility 60%, for four weeks.

a) Use the Brownian motion B_1 from the Problem 1 to construct one stock price path $S_1(t_j)$, $j = 0, 1, 2, 3, 4$. For the rest of this problem consider the path that you just constructed plus another path, S_2, given below:

j	0	1	2	3	4
t_j	0	1/52	1/26	3/52	1/13
$S_2(t_j)$	45.00	47.23	48.99	52.04	50.69

Answer: The geometric mean Brownian motion model for a stock price is

$$S(t) = S(0)\mathrm{e}^{(r-\sigma^2/2)t+\sigma B(t)}, \qquad t \geq 0.$$

This gives us

j	0	1	2	3	4
t_j	0	1/52	1/26	3/52	1/13
$S_2(t_j)$	45.00	47.23	48.99	52.04	50.69
$B_1(t_j)$	0	−0.1804	−0.0552	−0.0599	0.1300
$S_1(t_j)$	45.00	40.25	43.25	42.9885	48.02

b) For S_1 and S_2, what are the discounted payoffs of a *lookback call* option that expires four weeks from now? Are these *payoffs* less than, greater than, or equal to the discounted payoffs of a *European call* option with a strike price of \$45? Is the *price* of the lookback call option less than, greater than, or equal to the price of the European call option?

Answer: For the first stock path the discounted payoff is

$$[\max(S_1(1/13) - \min_{j=0,\ldots,4} S_1(t_j), 0)]\mathrm{e}^{-r/13} = [\max(48.02 - 40.25, 0)]\mathrm{e}^{-0.01/13} = 7.76$$

This payoff is greater than the European call payoff because the minimum stock price is less than the strike price. For the second stock path the discounted payoff is

$$[\max(S_2(1/13) - \min_{j=0,\ldots,4} S_2(t_j), 0)]\mathrm{e}^{-r/13} = [\max(50.69 - 45.00, 0)]\mathrm{e}^{-0.01/13} = 5.68$$

This payoff is the same as the European call payoff because the minimum stock price is the same as the strike price. The price of the lookback call is greater than the price of the European call because the payoff of the former is never less than and sometimes greater than the payoff of the latter.

c) Consider an *American put* option with a strike price of \$45. For each stock path, S_1 and S_2, will this option definitely have a zero payoff, definitely have a positive payoff, or only possibly have a positive payoff? Explain why based on the information provided.

Answer: The first stock path, S_1, dips below the strike price but ends up above the strike price at the time of expiry. It is not clear whether this stock path dips below the exercise boundary. Therefore it may possibly have a positive payoff, but not definitely.
The second stock path, S_2, always lies above the exercise boundary and will definitely have a zero payoff.

3. (30 points)
Let Y be a random variable whose mean, $\mu = \mathbb{E}(Y)$, you wish to estimate. Let X be a random variable with mean 5. Suppose that $\mathrm{var}(X) = 4$ and $\mathrm{var}(Y) = 6$. Furthermore, suppose that you are able to draw IID vector samples (X_i, Y_i), where $\mathrm{cov}(X_i, Y_i) = 3$. How many such samples do you need to obtain μ with a root mean square absolute error of 0.01 in the most efficient way? What is your estimator for μ?

Answer: We will use control variates. Let

$$Y_{\mathrm{CV}} = Y + \beta(5 - X), \qquad \hat{\mu}_{\mathrm{CV},n} = \frac{1}{n}\sum_{i=1}^{n} Y_{\mathrm{CV},i}, \quad Y_{\mathrm{CV},1}, Y_{\mathrm{CV},2}, \ldots \ \mathit{IID}$$

Note that $\hat{\mu}_{\mathrm{CV},n}$ *is unbiased because* $\mathbb{E}[Y_{\mathrm{CV}}] = Y + \beta(5 - X)] = \mathbb{E}(Y) + 0 = \mu$. *Moreover,*

$$\operatorname{var}(Y_{\mathrm{CV}}) = \operatorname{var}(Y) + \beta^2 \operatorname{var}(X) - 2\beta \operatorname{cov}(X, Y) = 6 + 4\beta^2 - 6\beta.$$

Choosing $\beta = 3/4$ *minimizes this variance and results in*

$$\operatorname{var}(Y_{\mathrm{CV}}) = 6 + \frac{9}{4} - \frac{9}{2} = \frac{15}{4}, \qquad Y_{\mathrm{CV}} = Y + \frac{3}{4}(5 - X)$$

Therefore, $\hat{\mu}_{\mathrm{CV},n}$ *has a root mean square error of* $\sqrt{\operatorname{var}(Y_{\mathrm{CV}})/n} = \sqrt{15/(4n)}$. *Choosing*

$$n = \left\lceil \frac{15}{4 \times (0.01)^2} \right\rceil = 37\,500$$

gives a root mean square absolute error of 0.01. *Using simple Monte Carlo requires*

$$n = \left\lceil \frac{6}{(0.01)^2} \right\rceil = 60\,000.$$

4. (15 points)
Consider the problem of evaluating the multidimensional integral

$$\int_{\mathbb{R}^d} \cos(x_1 + \cdots + x_d) \mathrm{e}^{-x_1^2 - \cdots - x_d^2} \, \mathrm{d}\boldsymbol{x}$$

by casting it as $\mathbb{E}[f(\boldsymbol{Z})]$ for standard Gaussian $\boldsymbol{Z}$, i.e, $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \mathsf{I}_d)$. What would be f in this case? Is f a bounded function or not?

Answer: If $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \mathsf{I}_d)$, *then*

$$\mathbb{E}[f(\boldsymbol{Z})] = \int_{\mathbb{R}^d} f(\boldsymbol{z}) \frac{\mathrm{e}^{-(z_1^2 + \cdots + z_d^2)/2}}{\sqrt{(2\pi)^d}} \, \mathrm{d}\boldsymbol{z}$$

So we need

$$f(\boldsymbol{z}) \frac{\mathrm{e}^{-(z_1^2 + \cdots + z_d^2)/2}}{\sqrt{(2\pi)^d}} = \cos(z_1 + \cdots + z_d) \mathrm{e}^{-z_1^2 - \cdots - z_d^2}$$

$$\begin{aligned} f(\boldsymbol{z}) &= \sqrt{(2\pi)^d} \mathrm{e}^{(z_1^2 + \cdots + z_d^2)/2} \cos(z_1 + \cdots + z_d) \mathrm{e}^{-z_1^2 - \cdots - z_d^2} \\ &= \sqrt{(2\pi)^d} \cos(z_1 + \cdots + z_d) \mathrm{e}^{-(z_1^2 + \cdots + z_d^2)/2}. \end{aligned}$$

Yes, f *is bounded.*