MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test has FOUR questions for a total of* 100 *points possible. You should attempt them all.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

Below are two independent Brownian motions that will be useful in the problems below.

time t	0.0625	0.1250	0.1875	0.2500
$B_1(t)$	-0.0973	-0.4252	0.0542	-0.0428
$B_2(t)$	-0.1811	-0.5147	-0.4528	-0.9960

The first three problems consider a stock whose price is modeled by a geometric Brownian motion. The price today is \$50, the volatility is σ year$^{-1/2}$, and the risk-free interest rate is 2% per year. The time frame is the next 3 months.

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1. (15 points)

a) What is the expected stock price 3 months from now to the nearest penny? How does your answer depend on σ?

Answer: The geometric Brownian motion model says that

$$S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma B(t)} = 50e^{(0.02-\sigma^2/2)t+\sigma B(t)}.$$

For the geometric Brownian motion we have the expected value of the stock is the same as the value of a zero volatility investment: $\mathbb{E}[S(t)] = S(0)e^{rt}$ *so* $\mathbb{E}[S(0.25)] = 50e^{0.02\times 0.25} =$ \$50.25.

b) What is the maximum possible value of the stock price that the stock may achieve in the next three months assuming positive volatility? Zero volatility?

Answer: Since $B(t)$ may be arbitrarily large, the maximum price that the stock price may achieve in the next three months is infinite for positive volatility and \$50.25 *for zero volatility.*

2. (35 points)
For this problem and the next problem assume that $\sigma = 50\%$ and that the strike price is \$50.

a) Construct two independent stock price paths.

Answer: Using the formula above we get

time t	0.0625	0.1250	0.1875	0.2500
$S_1(t)$	47.32	39.90	50.37	47.67
$S_2(t)$	45.37	38.15	39.09	29.60

b) Compute the *discounted* option payoffs to the nearest penny for these paths for the following options:

i. European put,

ii. Asian arithmetic mean put, and

iii. lookback put,

Answer: Noting that $K = 50$ *and* $rT = 0.005$*, the payoffs are as follows:*

i	*European put* $\max(50 - S_i(0.25), 0)\mathrm{e}^{-0.005}$
1	2.32
2	20.30

i	*Asian arithmetic mean put* $\max(50 - 0.25[S_i(0.0625) + \cdots + S_i(0.25)], 0)\mathrm{e}^{-0.005}$
1	3.67
2	11.89

i	*lookback put* $(\max_{0\le t\le 0.25} S_i(t) - S_i(0.25))\mathrm{e}^{-0.005}$
1	2.69
2	20.30

c) Is it ever possible for an Arithmetic mean put to have a higher discounted payoff than a European put? Explain why or why not.

Answer: It is possible when the stock price dips down for awhile and then climbs back up at the end, as for $i = 1$.

d) Is the American put payoff *zero* for either of the two paths that you have constructed, following an optimal exercise strategy? Explain why or why not.

Answer: No because the final stock price is in the money. If the option is not exercised until the expiry time, it will be in the money. If it is exercised earlier, it must be in the money.

3. (25 points)
Continue under the assumptions of the previous problem.

a) The actual price of the European put option with the above parameters is \$39.75. If the European put is used as a control variate to price the lookback put with $\beta = 0.6$, what are values of two the modified payoffs what would be used to price the lookback put? Why is it reasonable for β to be positive rather than negative?

Answer: The two modified payoffs would be

i	*Euro. put*	*lookback put*	$\textit{lookback}_i + 0.6(39.75 - \textit{Euro}_i)$
1	2.3200	2.6900	25.1480
2	20.3000	20.3000	31.9700

If the European put has a positive payoff, then the lookback put should have a positive payoff also. The two payoffs are positively correlated, so β should be positive.

b) Construct stock paths corresponding to two antithetic Brownian motions based on the above Brownian motions. What would the corresponding European payoffs be? What would be the two new random numbers for pricing the European option with antithetic variates be if you combined the original European payoffs and the antithetic payoffs?

Answer: To get the antithetic variates we choose $\widehat{B}(t) = -B(t)$ and use the formula

$$\widehat{S}(t) = S(0)\mathrm{e}^{(r-\sigma^2/2)t+\sigma\widehat{B}(t)} = 50\mathrm{e}^{(0.02-\sigma^2/2)t-\sigma B(t)}.$$

Since we only need European payoffs we only need to look at $t = 0.25$. We compute the stock prices at that time, and then the discounted European put payoffs. Then we take their average of the payoffs from the original stock price paths and the antithetic paths.

i	$S_i(0.25)$	$\hat{S}_i(0.25)$	$\textit{Euro}_i$	$\widehat{\textit{Euro}_i}$	$\frac{1}{2}[\textit{Euro}_i + \widehat{\textit{Euro}_i}]$
1	47.6700	49.7600	2.3200	0.2400	1.2800
2	29.6000	80.1400	20.3000	0.0000	10.1500

4. (25 points)
Consider the following multidimensional integral

$$I = \int_{[0,\infty)^d} f(\boldsymbol{x})\mathrm{e}^{-x_1-\cdots-x_d}\,\mathrm{d}\boldsymbol{x},$$

where f is some known function.

a) Suppose that you wish to approximate I by

$$\hat{I} = c\sum_{i=1}^{n} f(\boldsymbol{X}_i), \qquad \boldsymbol{X}_i \text{ IID}.$$

What should the value of c be and what should be the distribution of $\boldsymbol{X}$?

Answer: Note that $\varrho(x) = \mathrm{e}^{-x}$, $x \geq 0$ is the PDF of the exponential distribution with parameter 1, and $\int_0^\infty \varrho(x)\,\mathrm{d}x = 1$. So $c = 1/n$ and

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{X}_i), \qquad \boldsymbol{X}_{i,j} \textit{ IID Exponential}(1).$$

b) Suppose that you wish to approximate I by

$$\tilde{I} = \sum_{i=1}^{n} f(2\boldsymbol{X}_i) w(\boldsymbol{X}_i), \qquad \boldsymbol{X}_i \text{ IID the same distribution as in part a).}$$

What should $w(\boldsymbol{x})$ be?

Answer: Perform a change of variable, $\boldsymbol{x} = 2\boldsymbol{z}$, so

$$I = \int_{[0,\infty)^d} f(2\boldsymbol{z}) \mathrm{e}^{-2z_1 - \cdots - 2z_d} \, 2^d \mathrm{d}\boldsymbol{z} = \int_{[0,\infty)^d} f(2\boldsymbol{z}) 2^d \mathrm{e}^{-z_1 - \cdots - z_d} \mathrm{e}^{-z_1 - \cdots - z_d} \, \mathrm{d}\boldsymbol{z} \approx \tilde{I}$$

where $w(\boldsymbol{z}) = n^{-1} 2^d \mathrm{e}^{-z_1 - \cdots - z_d}$.