# MATH 565 Monte Carlo Methods in Finance

**Fred J. Hickernell** **Make-Up Test 2** **Monday, November 16, 2015**

*Instructions:*

i. *This test has FOUR questions for a total of* 100 *points possible. You should attempt them all.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

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The following are independent and identically distributed (IID) random variables from two different distributions. You will use some of these to solve some of the problems below.

| | | | | |
|---|---|---|---|---|
| Standard uniform, $\mathcal{U}[0,1]$ | 0.6780 | 0.5130 | 0.6237 | 0.4771 |
| Standard normal, $\mathcal{N}(0,1)$ | $-0.1269$ | 2.6097 | 2.7358 | 1.6672 |

The first three problems consider a stock whose price is modeled by a geometric Brownian motion. The price today is \$20, the volatility is 50% year$^{-1/2}$, and the risk-free interest rate is 2% per year. The time frame is the next 4 weeks, and the stock is monitored *weekly*.

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1. (30 points)

   a) Construct one stock price path using the random numbers above.

   *Answer: The geometric Brownian motion model says that*

   $$S(t) = S(0)\mathrm{e}^{(r-\sigma^2/2)t+\sigma B(t)} = 20\mathrm{e}^{(-0.1050)t+0.5B(t)}.$$

   *We choose $t$ values with an increment of* $1/52$. *We will construct a Brownian motion using time differencing and the* $\mathcal{N}(0,1)$ *numbers above. Then we use the above formula*

| *time t* | $1/52$ | $1/26$ | $3/52$ |
|---|---|---|---|
| $B(t)$ | $\sqrt{1/52}(-0.1269)$ $= -0.0176$ | $-0.0176 + \sqrt{1/52}(2.6097)$ $= 0.3443$ | $0.3443 + \sqrt{1/52}(2.7358)$ $= 0.7237$ |
| $S(t)$ | 19.78 | 23.66 | 28.55 |

| *time t* | $1/13$ |
|---|---|
| $B(t)$ | $0.7237 + \sqrt{1/52}(1.6672)$ $= 0.9549$ |
| $S(t)$ | 31.98 |

   b) Compute the *discounted* option payoffs to the nearest penny for this path for the following options, assuming a strike price of \$25:

      i. European call,

      ii. European put,

iii. Up and in barrier call, with a barrier of \$30, and

iv. Up and out barrier call, with a barrier of \$30.

*Answer: Noting that* $K = 25$ *and* $rT = 0.0015$, *the payoffs are as follows:*

| *European call* | *European put* |
|---|---|
| $\max(S(1/13) - 25, 0)e^{-0.0015}$ | $\max(25 - S(1/13), 0)e^{-0.0015}$ |
| $= 6.98$ | $= 0$ |

| *Up and in barrier call* | *Up and out barrier call* |
|---|---|
| *crossed barrier* | *crossed barrier* |
| 6.98 | 0 |

c) Of the three call options mentioned in b), which will have the highest price?

*Answer: The European call has the highest price because its payoff is greater than or equal to each of the barrier call options.*

2. (20 points)
Now consider an American put option for a stock under the same assumptions of the previous problem. Here is the exercise boundary and two possible stock price paths:

| time $t$ | 0.0192 | 0.0385 | 0.0577 | 0.0769 |
|---|---|---|---|---|
| exercise boundary | 19.76 | 20.24 | 20.58 | ??.?? |
| $S_1(t)$ | 21.96 | 22.73 | 22.68 | 23.89 |
| $S_2(t)$ | 18.67 | 19.73 | 19.70 | 18.13 |

Assume that the exercise boundary is below \$20.00 at time 0.

a) What is the exercise boundary at time 1/13?

*Answer: At the final time the exercise boundary is the strike price,* \$25.

b) When is each of these paths exercised and at what discounted payoff?

*Answer: The first path is exercised only at the final time, because that is the first time that the path crosses the exercise boundary. The discounted payoff is*

$$(25 - 23.89)e^{-0.02\times 1/13} = \$1.11.$$

*The second path is exercised only at the first nonzero time, because that is the first time that the path crosses the exercise boundary. The discounted payoff is*

$$(25 - 18.67)e^{-0.02\times 1/52} = \$6.33.$$

c) Do these paths achieve the maximum possible payoff if one could see into the future at each time?

*Answer: Neither path achieves the best possible payoff assuming future sight. If one knows the future, then the first path should be exercised at time* $1/52$ *and the second path should be exercised at time* $1/13$.

3. (25 points)
Consider a random variable $Y$ for which you wish to find $\mu = \mathbb{E}(Y)$ and another random variable $\widehat{Y}$, which has the *same mean.* Suppose that you are able to estimate $\text{var}(Y)$, $\text{cov}(Y, \widehat{Y})$, and $\text{var}(\widehat{Y})$ well. If you construct an estimator for $\mu$ of the form

$$\tilde{\mu}_n = \frac{1}{n}\sum_{i=1}^{n}[\theta Y_i + (1-\theta)\widehat{Y}_i], \qquad (Y_i, \widehat{Y}_i) \overset{\text{IID}}{\sim} (Y, \widehat{Y}),$$

how should $\theta$ be chosen as a function of $\text{var}(Y)$, $\text{cov}(Y, \widehat{Y})$, and $\text{var}(\widehat{Y})$ to minimize $\text{var}(\tilde{\mu}_n)$?

*Answer:*

$$\begin{aligned}
\text{var}(\tilde{\mu}_n) &= \frac{1}{n^2}\sum_{i=1}^{n}\text{var}[\theta Y_i + (1-\theta)\widehat{Y}_i] = \frac{\text{var}[\theta Y + (1-\theta)\widehat{Y}]}{n} \\
\text{var}[\theta Y + (1-\theta)\widehat{Y}] &= \theta^2\,\text{var}(Y) + 2\theta(1-\theta)\,\text{cov}(Y, \widehat{Y}) + (1-\theta)^2\,\text{var}(\widehat{Y}) \\
&= \theta^2[\text{var}(Y) - 2\,\text{cov}(Y, \widehat{Y}) + \text{var}(\widehat{Y})] - 2\theta[\text{var}(\widehat{Y}) - \text{cov}(Y, \widehat{Y})] + \text{var}(\widehat{Y})
\end{aligned}$$

*Optimizing this quadratic tells us that the minimum is obtained for*

$$\theta = \frac{\text{var}(\widehat{Y}) - \text{cov}(Y, \widehat{Y})}{\text{var}(Y) - 2\,\text{cov}(Y, \widehat{Y}) + \text{var}(\widehat{Y})}$$

*If* $\text{var}(Y) = \text{var}(\widehat{Y})$*, as in the case of antithetic variates, then the optimal* $\theta$ *is* $1/2$.

4. (25 points)
Consider the following multidimensional integral

$$I = \int_{[0,1]^4} \sin(x_1^2 + \cdots + x_4^2)\,\mathrm{d}\boldsymbol{x}.$$

You decide to approximate this integral using a Monte Carlo method that draws IID samples of the random variable $\boldsymbol{W} = (W_1, \ldots, W_4) \in [0,1]^4$, where the $W_j$ are IID with probability density function $\varrho$. Moreover, $\varrho(w)$ is proportional to $1 + w$ for $w \in [0,1]$.

a) The Monte Carlo estimator will then be

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} g(\boldsymbol{W}_i)$$

for what function $g : [0,1]^4 \to \mathbb{R}$?

*Answer: Since* $\varrho(w) = c(1+w)$, $0 \le w \le 1$, *then* $1 = c\int_0^1 (1+w)\,\mathrm{d}w = c(1+1/2)$, *and so* $c = 2/3$, *and the CDF is* $F(w) = \frac{2}{3}(w + w^2/2)$. *Thus,*

$$g(\boldsymbol{x}) = \frac{\sin(x_1^2 + \cdots + x_4^2)}{\varrho(x_1)\cdots\varrho(x_4)} = \frac{3^4 \sin(x_1^2 + \cdots + x_4^2)}{2^4(1+x_1)(1+x_2)(1+x_3)(1+x_4)}.$$

b) Compute an instance of $\boldsymbol{W}_1$ using the random numbers above.

*Answer: We use the inverse CDF*

$$u = \frac{2}{3}(w + w^2/2) \iff \frac{w^2}{2} + w - \frac{3u}{2} = 0 \iff w^2 + 2w - 3u = 0 \iff w = -1 + \sqrt{1+3u}$$

*This gives*

| | | | | |
|---|---|---|---|---|
| $U_j \sim \mathcal{U}[0,1]$ | 0.6780 | 0.5130 | 0.6237 | 0.4771 |
| $\boldsymbol{W} = (W_1, \ldots, W_4), W_j = -1 + \sqrt{1 + 3U_j}$ | 0.7418 | 0.5934 | 0.6944 | 0.5593 |