MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test has FOUR questions, each worth* 35 *points. You my choose which problems that you wish to attempt. All will be graded. The maximum possible score that you can earn is capped at* 100 *points, which will be considered a perfect score.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (35 points)
Consider the stochastic differential equation for the stock price that takes the form

$$\mathrm{d}\log(S(t)) = (-\sigma^2/2)\mathrm{d}t + \sigma\mathrm{d}B(t), \quad t \ge 0, \qquad S(0) = 50,$$

where B is a Brownian motion. Here we assume that the interest rate is zero and remember that log means the natural logarithm.

a) The geometric Brownian motion model for stock prices assumes that σ is a constant. Suppose that $\sigma = 0.4$, and that you generate a Brownian motion path with values

time t	1/12	1/6	1/4
Brownian motion $B(t)$	-0.1123	-0.4910	0.0626

What is $S(1/6)$ under this model?

Answer: The solution to the stochastic differential equation under constant volatility is

$$\int_0^t \mathrm{d}\log(S(s)) = \int_0^t \left[(-\sigma^2/2)\mathrm{d}s + \sigma\mathrm{d}B(s)\right],$$
$$\log(S(t)/S(0)) = \log(S(t)) - \log(S(0)) = (-\sigma^2/2)t + \sigma B(t),$$
$$S(t) = S(0)\exp\bigl((-\sigma^2/2)t + \sigma B(t)\bigr).$$

So,

$$S(1/6) = 50\exp\bigl((-0.4^2/2)(1/6) + 0.4(-0.4910)\bigr) = \$40.54.$$

b) Now assume that the stochastic differential equation above still holds, but that the volatility has a *skew*, in particular,

$$\sigma(S) = 0.4 + 0.2\left(\frac{S}{40} - 1\right).$$

Use the Brownian motion above to approximate $S(1/6)$ under this skew model.

Answer: Since σ is not constant, we cannot solve the stochastic differential equation exactly. However, it may be solved approximately:

$$\begin{aligned}
\int_t^{t+\Delta} \mathrm{d}\log(S(s)) &= \int_t^{t+\Delta} \big[(-\sigma(S)^2/2)\mathrm{d}s + \sigma(S)\mathrm{d}B(s)\big],\\
\log(S(t+\Delta)) - \log(S(t)) &\approx -\sigma(S(t))^2/2)\Delta + \sigma(S(t))[B(t+\Delta) - B(t)],\\
\log(S(t+\Delta)/S(t)) &\approx -\sigma(S(t))^2/2)\Delta + \sigma(S(t))[B(t+\Delta) - B(t)],\\
S(t+\Delta) &\approx S(t)\exp\big((-\sigma(S(t))^2/2)\Delta + \sigma(S(t))[B(t+\Delta) - B(t)]\big).
\end{aligned}$$

This means that

$$\begin{aligned}
\sigma(S(0)) &= 0.4 + 0.2\left(\frac{S(0)}{40} - 1\right) = 0.45\\
S(1/12) &\approx S(0)\exp\big((-\sigma(S(0))^2/2)\Delta + \sigma(S(0))[B(1/12) - B(0)]\big),\\
&\approx 50\exp\big((-0.45^2/2)(1/12) + 0.45[-0.1123 - 0]\big) = \$47.14,\\
\sigma(S(1/12)) &= 0.4 + 0.2\left(\frac{S(1/12)}{40} - 1\right) = 0.4317,\\
S(1/6) &\approx S(1/12)\exp\big((-\sigma(S(1/12))^2/2)\Delta + \sigma(S(1/12))[B(1/6) - B(1/12)]\big),\\
&\approx 46.35\exp\big((-0.4317^2/2)(1/12) + 0.4317[-0.4910 - (-0.1123)]\big) = \$39.65.
\end{aligned}$$

2. (35 points)
Suppose that one generates IID random vectors, (Y_i, X_i), $i = 1, \ldots, n$ for the purpose of estimating $\mu = \mathbb{E}(Y_1)$. Here, $\mu_X = \mathbb{E}(X_1)$ is known. Given this information, you decide to estimate μ by the control variate estimator $\hat{\mu}_{\mathrm{CV},n}$, where

$$\hat{\mu}_{\mathrm{CV},n} = \frac{1}{n}\sum_{i=1}^{n} Y_{\mathrm{CV},i}, \qquad Y_{\mathrm{CV},i} = Y_i + \hat{\beta}(\mu_X - X_i), \quad i = 1, \ldots, n.$$

You have computed the following using a rather large n:

$$\begin{aligned}
\widehat{\sigma}_{YY} &= \text{ the sample } \textit{variance} \text{ of the } Y_i, \quad \widehat{\sigma}_{YY} = 4\\
\widehat{\sigma}_{XX} &= \text{ the sample } \textit{variance} \text{ of the } X_i, \quad \widehat{\sigma}_{XX} = 2\\
\widehat{\sigma}_{YX} &= \text{ the sample } \textit{covariance} \text{ of the } Y_i \text{ and } X_i\\
\hat{\beta} &= 1
\end{aligned}$$

a) Are X_1 and Y_1 likely identically distributed?

Answer: No. They seem to have different variances.

b) Are X_1 and Y_1 likely independent?

Answer: Since $\hat{\beta} = \widehat{\sigma}_{YX}/\widehat{\sigma}_{XX}$ for control variates, $\widehat{\sigma}_{YX} = \hat{\beta}\widehat{\sigma}_{XX} = 2$. Since this is nonzero, X_1 and Y_1 are likely dependent.

c) If it takes $n = 10^5$ samples to estimate μ satisfactorily *without* control variates, about how large should n be to estimate μ satisfactorily *with* control variates?

Answer: Let $\mu_{Z,n} = n^{-1}\sum_{i=1}^{n} Z_i$ *where the* Z_i *are IID. Whether we look at root mean square error or the width of a confidence interval, both are proportional to* $\mathrm{std}(Z)/\sqrt{n}$*. Thus to meet a fixed error tolerance, we need* n *to be proportional to* $\mathrm{var}(Z)$*. Since*

$$\mathrm{var}(Y_{\mathrm{CV},1}) = \mathrm{var}(Y_1)[1 - \mathrm{corr}^2(Y, X)],$$

it follows that we can meet a fixed error tolerance, using control variates with only $1 - \mathrm{corr}^2(Y, X)$ *times the original sample size,* $n = 10^5$*.*

Note that

$$\mathrm{corr}^2(Y, X) = \frac{\mathrm{cov}^2(Y, X)}{\mathrm{var}(X)\,\mathrm{var}(Y)} \approx \frac{\widehat{\sigma}^2_{YX}}{\widehat{\sigma}_{XX}\widehat{\sigma}_{YY}} = \frac{2^2}{2 \times 4} = \frac{1}{2}.$$

So, only about 5×10^4 *samples are needed to estimate* μ *using control variates.*

3. (35 points)
Consider two functions defined on the domain $[-1, 1]$:

$$f_1(x) = 3x^2, \qquad f_2(x) = 1 - 4x^3.$$

a) Assuming $X \sim \mathcal{U}[-1, 1]$, compute by hand $\mu_j = \mathbb{E}[f_j(X)]$ for $j = 1, 2$.

Answer:

$$\mu_1 = \mathbb{E}[f_1(X)] = \int_{-1}^{1} f_1(x)\frac{1}{2}\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} 3x^2\,\mathrm{d}x = \frac{1}{2}x^3\Big|_{-1}^{1} = 1,$$
$$\mu_2 = \mathbb{E}[f_2(X)] = \int_{-1}^{1} f_2(x)\frac{1}{2}\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} 1 - 4x^3\,\mathrm{d}x = \frac{1}{2}(x - x^4)\Big|_{-1}^{1} = 1.$$

b) If $X_1 = -0.4$, what are the estimates of μ_j, $j = 1, 2$ based on *one sample using antithetic variates*?

Answer: For $X_1 = -0.4$*, the antithetic value is* $\widehat{X}_1 = -X_1 = 0.4$*. Then*

$$f_1(X_1) = f_1(-0.4) = 0.48 = f_1(0.4) = f_1(\widehat{X}_1), \quad \hat{\mu}_{1,1} = \frac{1}{2}[f_1(X_1) + f_1(\widehat{X}_1)] = 0.48,$$
$$f_2(X_1) = f_2(-0.4) = 1 + 0.256 = 1.256, \quad f_2(\widehat{X}_1) = f_2(0.4) = 1 - 0.256 = 0.744$$
$$\hat{\mu}_{2,1} = \frac{1}{2}[f_2(X_1) + f_2(\widehat{X}_1)] = 1.$$

c) For which function, f_1 or f_2, does the antithetic estimate give a better approximation to the true mean? What about a function makes the antithetic approximation more or less effective?

Answer: For f_2 *the approximation is exact. This is because, apart from its constant part,* f_2 *is antisymmetric, i.e.,* $f_2(-x) - 1 = -(f_2(x) - 1)$*. On the other hand,* f_1 *is* *symmetric, i.e.,* $f_1(-x) = f_1(x)$. *So, the values of* f_1 *at* X_1 *and* $\widehat{X}_1$ *are the same, and no extra information is gained from* $f_1(\widehat{X}_1)$. *Thus the approximation of* μ_1 *using antithetic variates is poorer than the approximation of* μ_2 *using antithetic variates.*

In general, the variance of an antithetic variates Monte Carlo approximation is $[1 + \operatorname{corr}(Y, \widehat{Y})]$ *times the variance of an IID Monte Carlo approximation. Thus, situations with negative* $\operatorname{corr}(Y, \widehat{Y})$ *are preferred. For,* f_1, *we have* $\operatorname{corr}(Y, \widehat{Y}) = 1$, *while for* f_2, *we have* $\operatorname{corr}(Y, \widehat{Y}) = -1$.

4. (35 points)
We want to estimate $\mu = \int_0^1 f(x)\,\mathrm{d}x$. For $i = 1, \ldots, n$, $n > 1$, define $Y_i = f\big((i - U_i)/n\big)$, where $U_1, \ldots, U_n \overset{\text{IID}}{\sim} \mathcal{U}[0, 1]$.

a) Is Y_1 an unbiased estimate of μ?

Answer: Since

$$\mathbb{E}(Y_i) = \int_0^1 f\big((i-u)/n\big)\,\mathrm{d}u \overset{t=(i-u)/n}{=} \int_{i/n}^{(i-1)/n} f(t)\,(-n)\,\mathrm{d}t = n \int_{(i-1)/n}^{i/n} f(t)\,\mathrm{d}t \neq \int_0^1 f(x)\,\mathrm{d}x = \mu,$$

Y_1, *or any* Y_i *is not an unbiased estimate of* μ.

b) Is $\hat{\mu}_{S,n} = n^{-1} \sum_{i=1}^n Y_i$ an unbiased estimate of μ?

Answer: Using the work in part a),

$$\mathbb{E}(\hat{\mu}_{S,n}) = \frac{1}{n} \sum_{i=1}^n E(Y_i) = \frac{1}{n} \sum_{i=1}^n n \int_{(i-1)/n}^{i/n} f(t)\,\mathrm{d}t = \int_0^1 f(x)\,\mathrm{d}x = \mu,$$

so, $\hat{\mu}_{S,n}$ *is an unbiased estimate of* μ.

c) Let J be a random variable distributed uniformly on $\{1, \ldots, n\}$ and independent of $U_1, \ldots, U_n$. Is Y_J an unbiased estimate of μ?

Answer: Using the work in part a),

$$\mathbb{E}(Y_J | J = i) = \mathbb{E}(Y_i) = n \int_{(i-1)/n}^{i/n} f(t)\,\mathrm{d}t.$$

So,

$$\mathbb{E}(Y_J) = \mathbb{E}_J\big[\mathbb{E}(Y_J|J)\big] = \sum_{i=1}^n \mathbb{E}(Y_J | J = i)\Pr(J = i) = \sum_{i=1}^n n \int_{(i-1)/n}^{i/n} f(t)\,\mathrm{d}t \times \frac{1}{n} = \int_0^1 f(x)\,\mathrm{d}x = \mu,$$

and Y_J *is an unbiased estimate of* μ.