MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test has THREE questions, worth a total of* 100 *points. Attempt as many as you can.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *(Programmable) calculators are allowed, but they must not have stored text.*

v. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (33 points)
The Vasicek model for interest rate fluctations is

$$\mathrm{d}r = \alpha(r_0 - r(t))\mathrm{d}t + \sigma \mathrm{d}B(t),$$

where r is the interest rate, t is time, r_0 is the long term interest rate, σ is the volatility of the interest rate, and B is a Brownian motion. Given $\alpha = 1$, $r_0 = 0.01$, $\sigma = 0.01$, and $r(1/4) = 0.015$, give either an exact value for $r(1/3)$ or a good approximation to $r(1/3)$. You may take $Z = -0.3890$ to be an instance of a $\mathcal{N}(0,1)$ random variable.

Answer: The exact solution is given by $\Delta = 1/12$,

$$r(1/3) = e^{-\alpha\Delta} r(1/4) + \left(1 - e^{-\alpha\Delta}\right) r_0 + \sigma\sqrt{\frac{1 - e^{-2\alpha\Delta}}{2\alpha}} Z = 0.0124.$$

An Euler method approximation is

$$r(1/3) \approx r(1/4) + \alpha(r_0 - r(1/4))\Delta + \sigma\sqrt{\Delta} Z = 0.0135.$$

2. (33 points)
Suppose that you are trying to improve a simple IID Monte Carlo estimate for $\mu = \mathbb{E}(Y)$ by using a control variate X with mean μ_X. Using the optimal control variate coefficient estimated from data, $\hat{\beta} = 0.3$, you find that the standard deviation of your control variate estimate for μ is $1/5$ of the standard deviation of the original simple IID Monte Carlo estimate for μ.

   a) If you needed 10^6 samples to reach your desired tolerance with the simple IID Monte Carlo estimate, how many samples will you need with the control variate estimate?

   Answer: The sample size is proportional to the variance, so you will only need $1/25$ *the original number of samples or* 4×10^4 *samples.*

   b) If you use as your control variate $Z = 10X$ instead of X, what would be the optimal coefficient now? What would be the number of samples required to meet your tolerance using this new control variate?

   Answer: Using Z *as control variate gives the same improvement in the variance of the Monte Carlo estimator and reduction in the necessary sample size as using* X*. The optimal coefficient now becomes* $0.3/10 = 0.03$.

c) If you use two control variates, X and $Z = 10X$, how much smaller will the variance for your Monte Carlo estimator be than if you use X alone?

Answer: The variance will be the same. Using X or Z alone, or using X and Z together, gives the same estimator, because Z and X are essentially the same.

3. (34 points)

You want to approximate the integral $\mu = \int_{\mathbb{R}^2} f(\boldsymbol{x}) \exp(-(x_1^2 + x_2^2)/2) \, \mathrm{d}\boldsymbol{x}$ using a Monte Carlo algorithm.

a) Write a formula for $\hat{\mu}$, a Monte Carlo approximation to μ, using $X_1, \ldots, X_{10\,000} \overset{\text{IID}}{\sim} \mathcal{N}(0, 1)$.

Answer:

$$\begin{aligned}
\mu &= \int_{\mathbb{R}^2} f(\boldsymbol{x}) \exp(-(x_1^2 + x_2^2)/2) \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^2} (2\pi) f(\boldsymbol{x}) \frac{\exp(-(x_1^2 + x_2^2)/2)}{2\pi} \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} (2\pi) f(\boldsymbol{x}) \varrho(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}
\end{aligned}$$

where ϱ is the probability density function (PDF) for the $\mathcal{N}(0, 1)$ distribution. So,

$$\hat{\mu} = \frac{2\pi}{5\,000} \sum_{i=1}^{5\,000} f(X_{2i-1}, X_{2i}).$$

b) Write a formula for $\tilde{\mu}$, a Monte Carlo approximation to μ, using $Z_1, \ldots, Z_{10\,000} \overset{\text{IID}}{\sim} \mathcal{N}(0, 4)$. (These Z_i are independent of the X_i in the previous part.)

Answer: This is an example of importance sampling:

$$\begin{aligned}
\mu &= \int_{\mathbb{R}^2} f(\boldsymbol{x}) \exp(-(x_1^2 + x_2^2)/2) \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{x}) \frac{8\pi \exp(-(x_1^2 + x_2^2)/2)}{\exp(-(x_1^2 + x_2^2)/(2 \times 4))} \frac{\exp(-(x_1^2 + x_2^2)/(2 \times 4))}{2\pi \times 4} \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{x}) 8\pi \exp(-3(x_1^2 + x_2^2)/8) \, \tilde{\varrho}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}
\end{aligned}$$

where $\tilde{\varrho}$ is the probability density function (PDF) for the $\mathcal{N}(0, 4)$ distribution. So,

$$\tilde{\mu} = \frac{8\pi}{5\,000} \sum_{i=1}^{5\,000} f(Z_{2i-1}, Z_{2i}) \exp(-3(Z_{2i-1}^2 + Z_{2i}^2)/8).$$

c) You discover that the sample variance of the function values used to compute $\tilde{\mu}$ is only $1/4$ of the sample variance of the function values used to compute $\hat{\mu}$. For what value of θ would $(1 - \theta)\hat{\mu} + \theta\tilde{\mu}$ be the best possible estimate for μ?

Answer: Note that $(1-\theta)\hat{\mu}+\theta\tilde{\mu}$ *is unbiased. The variances of the function values carry over to the variances of the two Monte Carlo approximations, since they are independent:*

$$\begin{aligned}\mathrm{var}((1-\theta)\hat{\mu}+\theta\tilde{\mu}) &= (1-\theta)^2\,\mathrm{var}(\hat{\mu})+\theta^2\,\mathrm{var}(\tilde{\mu}) = [(1-\theta)^2+\theta^2/4]\,\mathrm{var}(\hat{\mu})\\ &= (1-2\theta+5\theta^2/4)\,\mathrm{var}(\hat{\mu}).\end{aligned}$$

This is minimized by taking $\theta = 4/5$*, so* $\hat{\mu}/5+4\tilde{\mu}/5$ *is the best estimator, and it has a variance of* $\mathrm{var}(\hat{\mu})/5$*.*