MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test has FOUR questions. Attempt them all. The maximum number of points is* 100.

ii. *The time allowed is 75 minutes.*

iii. *Keep at least* four *significant digits in your intermediate calculations and final answers.*

iv. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

v. *(Programmable) calculators are allowed, but they must not have stored text.*

vi. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (10 points)
Consider the three sequences of numbers that are *claimed* to be IID $\mathcal{U}[-1,1]$. Which one or more of these sequences *do not* look IID $\mathcal{U}[-1,1]$?

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
(a)	0.0086	0.7450	0.1590	0.2317	0.1489	0.3082	0.6583	0.9271
(b)	−0.0181	−0.2678	−0.4750	0.5970	0.9196	0.7675	−0.4288	0.5121
(c)	0.8108	−0.8108	0.0250	−0.0250	0.0660	−0.0660	0.1068	−0.1068

Answer: Sequence (a) has only positive numbers, which is unlikely since the X_i have equal probability of being negative or positive. Sequence (c) has alternating positive and negative numbers of the same magnitude, which is unlikely. In (c) it seems that X_{2i-1} and X_{2i} are perfectly negatively correlated. Sequence (b) has no suspicious features.

2. (30 points)
Generate a sequence of six IID numbers, $Y_1, \ldots, Y_6$, that satisfy the following *discrete* probability distribution:

y	1	2	3	4	5
$\mathbb{P}(Y=y)$	0.3000	0.3000	0.2000	0.1000	0.1000

by means of the following six IID $\mathcal{U}[0,1]$ random numbers:

X_1	X_2	X_3	X_4	X_5	X_6
0.5533	0.9328	0.0048	0.8717	0.0826	0.8112

Answer: First we compute the cumulative distribution function (CDF) for Y. We also compute the inverse CDF.

y	1	2	3	4	5
$\mathbb{P}(Y=y)$	0.3000	0.3000	0.2000	0.1000	0.1000
$F_Y(y)=\mathbb{P}(Y\le y)$	0.3000	0.6000	0.8000	0.9000	1.0000
$y=F_Y^{-1}(x)$ *for* $x\in$	$[0.0,0.3)$	$[0.3,0.6)$	$[0.6,0.8)$	$[0.8,0.9)$	$[0.9,1.0]$

Then we use the inverse CDF transformation to find Y_i in terms of X_i:

X_1	X_2	X_3	X_4	X_5	X_6
0.5533	0.9328	0.0048	0.8717	0.0826	0.8112
Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
2	5	1	4	1	4

3. (30 points)
Consider a stock whose price is modeled by a geometric Brownian motion and is monitored *monthly* for *three months*. The initial price is \$40, the volatility is 50%, and the interest rate is 2%. Let

Z_1	Z_2	Z_3
−1.4234	−1.1188	−0.7863

be IID $\mathcal{N}(0,1)$ variables.

a) Construct *one* stock price path for this stock.

Answer: Let $t_j = j/12$ for $j = 1, 2, 3$. We first construct the Brownian motion by

$$B(t_j) = B(t_{j-1}) + \sqrt{1/12} Z_j, \qquad j = 1, 2, 3.$$

Then we compute the stock price by

$$S(t_j) = S(0) \exp\big((r - \sigma^2/2)t_j + \sigma B(t_j)\big), \qquad j = 1, 2, 3, \quad \sigma = 0.5.$$

So,

j	0	1	2	3
t_j	0.0000	0.0833	0.1667	0.2500
$B(t_j)$	0.0000	−0.4109	−0.7339	−0.9609
$S(t_j)$	40.00	32.29	27.23	24.10

b) For the stock price path in a), what is the discounted payoff of a *European put* option with a strike price of \$50 that expires three months from now?

Answer: The payoff is $(\$50 - \$24.10)e^{-0.02\times 1/4} = \$25.77$.

c) For the stock price path in a), what is the discounted payoff of a *lookback put* option that expires three months from now?

Answer: Since the payoff is also $(\$40 - \$24.10)e^{-0.02\times 1/4} = \$15.82$, *since* \$40 *is the highest price along the path.*

d) For the stock price path in a), what is the discounted payoff of an *American put* option with a strike price of \$50 that expires three months from now? Assume that the exercise boundary is given as follows:

j	0	1	2	3
t_j	0.0000	0.0833	0.1667	0.2500
$b(t_j)$	30.00	31.00	35.00	??

Explain why this discounted American payoff is greater than or is less than the discounted European payoff in part b).

Answer: The stock price first falls below the exercise boundary for $t = 0.1667$*, so the payoff is* $(\$50 - \$27.23)\mathrm{e}^{-0.02\times 1/6} = \$22.69$*. The option is exercised early because the expected value of holding the option at time* $1/6$ *is less than the value of exercising then. But the value for this particular path is greater if the option is held. Unfortunately, the option holder cannot predict that.*

4. (30 points)
Let Y be a random variable for which you wish to find $\mu = \mathbb{E}(Y)$. A Monte Carlo estimate of μ is

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

Let X be another random variable for which $\mu_X = \mathbb{E}(X)$ is known.

a) Let $(Y_1, X_1), (Y_2, X_2), \ldots$ be IID random vectors, and let $Y_{\mathrm{CV},i} = Y_i - \beta(X_i - \mu_X)$. For which value(s) of β is

$$\hat{\mu}_{\mathrm{CV},n} = \frac{1}{n}\sum_{i=1}^{n} Y_{\mathrm{CV},i}$$

an unbiased estimator of μ?

Answer:

$$\mathbb{E}[Y_{CV,i}] = \mathbb{E}[Y_i - \beta(X_i - \mu_X)] = \mu, \text{ so } \mathbb{E}[\hat{\mu}_{CV,n}] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[Y_{CV,i}] = \mu$$

for all β*, and* $\hat{\mu}_{CV,n}$ *is an unbiased estimator of* μ *for all* β*.*

b) Under what condition on the correlation between X and Y is the root mean square error of $\hat{\mu}_{\mathrm{CV},n/2}$ no worse than the root mean square error of $\hat{\mu}_n$, assuming that β is chosen optimally?

Answer: For optimal β*,*

$$\mathrm{RMSE}(\hat{\mu}_n) = \sqrt{\frac{\mathrm{var}(Y)}{n}}, \qquad \mathrm{RMSE}(\hat{\mu}_{CV,n/2}) = \sqrt{\frac{\mathrm{var}(Y_{CV})}{n/2}} = \sqrt{\frac{2\,\mathrm{var}(Y_{CV})}{n}},$$
$$2\,\mathrm{var}(Y_{CV}) = 2\,\mathrm{var}(Y)[1 - \mathrm{corr}^2(Y,X)].$$

So, we need $2[1 - \mathrm{corr}^2(Y,X)] \le 1$ *to ensure that the root mean square error of* $\hat{\mu}_{CV,n/2}$ *no worse than the root mean square error of* $\hat{\mu}_n$*. This means that we need* $|\mathrm{corr}(Y,X)| \ge 1/\sqrt{2} \approx 0.7071$*.*