MATH 565 Monte Carlo Methods in Finance

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Instructions:

i. *This test has FOUR questions. Attempt them all. The maximum number of points is* 100.

ii. *The time allowed is 75 minutes.*

iii. *Keep at least* four *significant digits in your intermediate calculations and final answers.*

iv. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

v. *(Programmable) calculators are allowed, but they must not have stored text. No phones are allowed.*

vi. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

1. (20 points)
Let B be a Brownian motion, and let τ and t be two times satisfying $0 < \tau < t$. Use the basic assumptions about the Brownian motion to answer the following:

   a) What is $\mathbb{E}[\{B(t)\}^2]$?
   b) What is the correlation between $B(\tau)$ and $B(t) - B(\tau)$?
   c) What is the correlation between $B(t)$ and $B(t) - B(\tau)$?

Answer: We note that $\mathbb{E}[B(t)] = 0$ *and* $\operatorname{cov}\bigl(B(\tau), B(t)\bigr) = \min(\tau, t)$ *for* $0 \le \tau, t$. *Then*

$$\begin{aligned}
\mathbb{E}[\{B(t)\}^2] &= \operatorname{var}\bigl(B(t)\bigr) = \operatorname{cov}\bigl(B(t), B(t)\bigr) = t, \\
\operatorname{var}\bigl(B(t) - B(\tau)\bigr) &= \operatorname{var}\bigl(B(t)\bigr) - 2\operatorname{cov}\bigl(B(\tau), B(t)\bigr) + \operatorname{var}\bigl(B(\tau)\bigr) = t - 2\tau + \tau \\
&= t - \tau \\
\operatorname{corr}\bigl(B(\tau), B(t) - B(\tau)\bigr) &= \frac{\operatorname{cov}\bigl(B(\tau), B(t) - B(\tau)\bigr)}{\sqrt{\operatorname{var}\bigl(B(\tau)\bigr)\operatorname{var}\bigl(B(t) - B(\tau)\bigr)}} \\
&= \frac{\operatorname{cov}\bigl(B(\tau), B(t)\bigr) - \operatorname{cov}\bigl(B(\tau), B(\tau)\bigr)}{\sqrt{\tau(t-\tau)}} = \frac{\tau - \tau}{\sqrt{\tau(t-\tau)}} = 0 \\
\operatorname{corr}\bigl(B(t), B(t) - B(\tau)\bigr) &= \frac{\operatorname{cov}\bigl(B(t), B(t) - B(\tau)\bigr)}{\sqrt{\operatorname{var}\bigl(B(t)\bigr)\operatorname{var}\bigl(B(t) - B(\tau)\bigr)}} \\
&= \frac{\operatorname{cov}\bigl(B(t), B(t)\bigr) - \operatorname{cov}\bigl(B(t), B(\tau)\bigr)}{\sqrt{t(t-\tau)}} = \frac{t - \tau}{\sqrt{t(t-\tau)}} \\
&= \sqrt{\frac{t-\tau}{t}} = \sqrt{1 - \frac{\tau}{t}}
\end{aligned}$$

So how much we are changing in the future does not depend on where we are now, but how much we changed from the past does depend on where we are now and how long ago the past was.

2. (20 points)
A stock is monitored *monthly* for one quarter of a year. Its price is modeled by a geometric Brownian motion. The initial price is \$50. The interest rate is 1% year^{-1}. The volatility is 40% year$^{-1/2}$. Use the following IID Gaussian (normal) random numbers to produce one stock price path:

Z_1	Z_2	Z_3
1.6125	−0.5478	−0.8442

Answer: The stock price path is given by

$$\begin{aligned} S(j/12) &= S(0)\exp\big((r-\sigma^2/2)(j/12)+\sigma B(j/12)\big), \quad j=1,2,3 \\ &= 50\exp\big(-0.07(j/12)+0.4B(j/12)\big), \quad j=1,2,3 \end{aligned}$$

We need to compute the Brownian motion from the three normal random variables. We do this by noting that

$$B(j/12) = B((j-1)/12) + \sqrt{1/12}Z_j, \qquad j=1,2,3$$

Then we plug this into the formula above

t	0	1/12	1/6	1/4
$B(t)$	0	$\sqrt{1/12}Z_1$ $= 0.4655$	$0.4655+\sqrt{1/12}Z_2$ $= 0.3073$	$0.3073+\sqrt{1/12}Z_3$ $= 0.06365$
$S(t)$	50	59.88	55.88	50.40

3. (30 points)
Under the same assumptions as in the previous problem, four stock price paths are generated:

t	0	1/12	1/6	1/4
$S_1(t)$	50.0000	48.6600	50.2500	46.7200
$S_2(t)$	50.0000	52.9100	53.2900	50.2100
$S_3(t)$	50.0000	61.6500	49.3200	52.1400
$S_4(t)$	50.0000	60.7500	50.3700	49.0400

Use Monte Carlo with this small sample size to approximate the prices of the following two options:

a) A up-and-in call with a strike price of \$40 and barrier of \$60; and
b) A lookback call option.

Answer: First we compute the payoff for each path. The discounted up-and-in call payoff is

$$\max(S(1/4)-40,0)\exp(-0.01/4),$$

only when the stock path exceeds \$60 *at some point in time. The discounted lookback call path is*

$$\left(S(1/4) - \min_{t=0,\ldots,1/4} S(t)\right)\exp(-0.01/4).$$

Then we take the average of the discounted payoffs to get the price.

t	0	1/12	1/6	1/4	*Up-In Payoff*	*Lookback Payoff*
$S_1(t)$	50.0000	48.6600	50.2500	46.7200	0.000	0.000
$S_2(t)$	50.0000	52.9100	53.2900	50.2100	0.000	0.209
$S_3(t)$	50.0000	61.6500	49.3200	52.1400	12.110	2.813
$S_4(t)$	50.0000	60.7500	50.3700	49.0400	9.017	0.000
Price					5.282	0.756

4. (30 points)
To find $\mu = \mathbb{E}(Y)$, you wish to use the *control variate* X, which has population mean -0.5731. After generating 1000 pairs of data, $(Y_1, X_1), \ldots, (Y_{1000}, X_{1000})$, you compute the sample quantities

$$\begin{aligned}
\hat{\mu}_Y &= \text{the sample mean of } Y = 3.574 \\
\hat{\mu}_X &= \text{the sample mean of } X = -0.5561 \\
\hat{\sigma}_{YY} &= \text{the sample } \textit{variance} \text{ of } Y = 12.84 \\
\hat{\sigma}_{XX} &= \text{the sample } \textit{variance} \text{ of } X = 2.991 \\
\hat{\sigma}_{XY} &= \text{the sample } \textit{covariance} \text{ of } X \text{ and } Y = 4.841
\end{aligned}$$

a) What is your 99% confidence interval for μ based on the sampling of Y alone?

Answer:

$$\hat{\mu}_Y \pm \frac{2.58 \times 1.2 \times \sqrt{\hat{\sigma}_{YY}}}{\sqrt{1000}} = 3.574 \pm 0.3508$$

b) What is your 99% confidence interval for μ based on the control variate X?

Answer: For control variates we have $Y_{\text{CV}} = Y - \beta(X - \mu_X)$. The best β is

$$\hat{\beta} = \frac{\hat{\sigma}_{XY}}{\sigma_{XX}} = 1.619$$

so that the the sample mean of Y_{CV} is

$$\begin{aligned}
\hat{\mu}_{\text{CV}} &= \hat{\mu}_Y - \hat{\beta}(\hat{\mu}_X - \mu_X) = 3.574 - 1.619(-0.5561 + 0.5731) = 3.546 \\
\text{var}(Y_{\text{CV}}) &\approx \hat{\sigma}_{YY} - 2\hat{\beta}\hat{\sigma}_{XY} + \hat{\beta}^2\hat{\sigma}_{XX} = 5.005
\end{aligned}$$

and the width of the confidence interval using control variates is

$$\hat{\mu}_{\text{CV}} \pm \frac{2.58 \times 1.2 \times \sqrt{\text{var}(Y_{\text{CV}})}}{\sqrt{1000}} = 3.546 \pm 0.2190$$

c) What would be your 99% confidence interval for μ if you used the control variate $-X$ instead of X?

Answer: The confidence interval would be the same. The value of $\hat{\beta}$ would be minus the original.

d) With the information given above, is it possible to use Y as a control variate for estimating $\mathbb{E}(X)$? Why or why not?

Answer: No. We do not know the population mean of Y. Also, we already know the population of X so there is no need.