MATH 565 Monte Carlo Methods

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Instructions:

i. *This test has* ***THREE*** *questions. Attempt them all. The maximum number of points is* ***100****.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *No calculators or other devices are allowed. Phones must be placed in your bags under your desks or face down on your desks. Hands mut be on top of your desks.*

v. *Keep at least* three *significant digits in your intermediate calculations and final answers.*

vi. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

vii. *Off-site students may contact the instructor as directed by your syllabus.*

I understand these instructions and have not relied on any help for this exam beyond what is allowed, nor provided any help to anyone else beyond what is allowed.

______________________ *Signature* ______________________ *Date*

1. (30 points) For some $f : [0, 2] \times [0, \infty) \to \mathbb{R}$ you want to approximate

$$\mu = \int_0^\infty \int_0^2 f(x, y) \exp(-y) \, \mathrm{d}x\mathrm{d}y$$

by IID Monte Carlo. For each $\hat{\mu}$ below determine whether or not it is an *unbiased estimator* of μ for general f. Assume that the X_i and Y_i are independent of each other. [Note: an $\mathrm{Exp}(\lambda)$ random variable has a density defined as $\varrho(y) = \lambda \exp(-\lambda y)$ for non-negative y and zero for negative y.]

Answer: Note that in every case

$$\mathbb{E}[\hat{\mu}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=0}^{n-1} g(X_i, Y_i)\right] = \mathbb{E}[g(X, Y)]$$

a. (15 points)

$$\hat{\mu} = \frac{1}{n}\sum_{i=0}^{n-1} f(X_i, Y_i) \exp(-Y_i), \qquad X_i \overset{\text{IID}}{\sim} \mathcal{U}[0, 2], \; Y_i \overset{\text{IID}}{\sim} \mathrm{Exp}(1)$$

Answer: Since the density of the X_i is $1/2$ and the density of the Y_i is $y \mapsto \exp(-y)$,

$$\begin{aligned}
\mathbb{E}[\hat{\mu}] &= \mathbb{E}[f(X,Y)\exp(-Y)]), \qquad X_i \overset{\text{IID}}{\sim} \mathcal{U}[0,2],\ Y_i \overset{\text{IID}}{\sim} \mathit{Exp}(1) \\
&= \int_0^\infty \int_0^2 f(x,y)\exp(-y) \times \frac{1}{2}\exp(-y)\,\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty \int_0^2 \frac{1}{2} f(x,y)\exp(-2y))\,\mathrm{d}x\mathrm{d}y \neq \mu,
\end{aligned}$$

so the answer is, "No."

b. (15 points)

$$\hat{\mu} = \frac{1}{n}\sum_{i=0}^{n-1} f(X_i,Y_i)\exp(Y_i), \qquad X_i \overset{\text{IID}}{\sim} \mathcal{U}[0,2],\ Y_i \overset{\text{IID}}{\sim} \mathrm{Exp}(2)$$

Answer: In this case the density of the X_i is $1/2$ and the density of the Y_i is $y \mapsto 2\exp(-2y)$,

$$\begin{aligned}
\mathbb{E}[\hat{\mu}] &= \mathbb{E}[f(X,Y)\exp(Y)] \quad X \overset{\text{IID}}{\sim} \mathcal{U}[0,2],\ Y \overset{\text{IID}}{\sim} \mathrm{Exp}(2) \\
&= \int_0^\infty \int_0^2 f(x,y)\exp(y) \times \frac{1}{2} \times 2\exp(-2y)\,\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty \int_0^2 f(x,y)\exp(-y)\,\mathrm{d}x\mathrm{d}y = \mu,
\end{aligned}$$

so the answer is, "Yes."

One should understand importance sampling and variable transformations. One cannot write $f(x,y) = f(x)f(y)$ because $f(x,y)$ is not the density. When you take the expectation of $f(X,Y)$ you must take the integral of $f(x,y)$ times the probability density.

2. (40 points) Consider the table of proposed random variable values, Z_i, target density values $\varrho(Z_i)$, and uniform random variable values, $U_i \overset{\text{IID}}{\sim} \mathcal{U}[0,1]$:

i	0	1	2
Z_i	0.3	1.2	0.8
$\varrho(Z_i)$	0.9	0.5	0.7
U_i	0.3	0.9	0.1

Starting with $X_0 = 0.5$ and $\varrho(X_0) = 0.5$,

a. (20 points) Find X_1, X_2, and X_3 by the Metropolis algorithm?

Answer: At every step we check whether U_i is less than the likelihood ratio, $\varrho(Z_i)/\varrho(X_i)$. If so, we accept $X_{i+1} = Z_i$. Otherwise we keep $X_{i+1} = X_i$. These are the results

i	X_i	$\varrho(X_i)$	Z_i	$\varrho(Z_i)$	$\varrho(Z_i)/\varrho(X_i)$	U_i	$U_i \overset{?}{\le} \varrho(Z_i)/\varrho(X_i)$	X_{i+1}
0	*0.500*	*0.500*	*0.300*	*0.900*	*1.800*	*0.300*	*yes, accept*	*0.300*
1	*0.300*	*0.900*	*1.200*	*0.500*	*0.556*	*0.900*	*no, reject*	*0.300*
2	*0.300*	*0.900*	*0.800*	*0.700*	*0.778*	*0.100*	*yes, accept*	*0.800*

b. (5 points) What values of U_0 (if any) would change your decision on whether to accept Z_0 as X_1?

Answer: Impossible since the likelihood ratio is larger than one.

c. (5 points) What values of U_2 (if any) would change your decision on whether to accept Z_2 as X_3?

Answer: If $U_2 > 0.778$ *then we would reject* Z_2 *as* X_3.

d. (10 points) Explain what difference it makes in the Metropolis algorithm if ϱ is unnormalized, i.e. $c\varrho$ is the target density for some positive $c \neq 1$?

Answer: It makes no difference since the likelihood ratio is the same for unnormalized or normalized densities.

This problem follows an example in class. If the likelihood ratio is bigger than one, we must always accept since the uniform random decision variable can be no greater than one.

3. (30 points) Some special kernels, K, satisfy

$$\int_{\mathcal{X}} K(\boldsymbol{x}, \boldsymbol{z})\, \varrho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_{\mathcal{X}} K(\boldsymbol{x}, \boldsymbol{z})\, \mathrm{d}F(\boldsymbol{x}) = 1 \qquad \forall \boldsymbol{z},$$

where ϱ is the target density, and F is the target distribution.

a. (10 points) How does the formula for $D^2(F, \{\boldsymbol{z}\}_{i=0}^{n-1}, K)$, the squared discrepancy of a set of points $\{z_i\}_{i=0}^{n-1}$ in comparison to F, simplify in this case?

Answer: The formula for the squared discrepancy is

$$\begin{aligned} D^2\big(F, \{\boldsymbol{z}_i\}_{i=0}^{n-1}; K\big) &= \int_{\mathcal{X}\times\mathcal{X}} K(\boldsymbol{x}, \boldsymbol{z})\, \varrho(\boldsymbol{x})\varrho(\boldsymbol{z})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{z} - \frac{2}{n}\sum_{i=0}^{n-1} \int_{\mathcal{X}} K(\boldsymbol{x}, \boldsymbol{z}_i)\, \varrho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ &\qquad + \frac{1}{n^2}\sum_{i,j=0}^{n-1} K(\boldsymbol{z}_i, \boldsymbol{z}_j) \\ &= -1 + \frac{1}{n^2}\sum_{i,j=0}^{n-1} K(\boldsymbol{z}_i, \boldsymbol{z}_j) \end{aligned}$$

b. (20 points) If $\{z_i\}_{i=0}^{n-1}$ are IID points from the target distribution, then what is the mean squared discrepancy in terms of K and n?

Answer:

$$\mathbb{E}\big[D^2\big(F, \{\boldsymbol{z}_i\}_{i=0}^{n-1}; K\big)\big] = -1 + \frac{1}{n^2} \sum_{i,j=0}^{n-1} \mathbb{E}\big[K(\boldsymbol{z}_i, \boldsymbol{z}_j)\big]$$

There are two cases:

- *When* $i = j$ *(*n *out of* n^2 *times),* $\mathbb{E}\big[K(\boldsymbol{z}_i, \boldsymbol{z}_j)\big] = \mathbb{E}\big[K(\boldsymbol{z}_i, \boldsymbol{z}_i)\big] = \int_{\mathcal{X}} K(\boldsymbol{x}, \boldsymbol{x})\, \varrho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$.
- *When* $i \neq j$ *(*$n^2 - n$ *out of* n^2 *times),* $\mathbb{E}\big[K(\boldsymbol{z}_i, \boldsymbol{z}_j)\big] = \int_{\mathcal{X}\times\mathcal{X}} K(\boldsymbol{x}, \boldsymbol{z})\, \varrho(\boldsymbol{x})\varrho(\boldsymbol{z})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{z} = 1$.

Thus,

$$\begin{aligned}\mathbb{E}\big[D^2\big(F, \{\boldsymbol{z}_i\}_{i=0}^{n-1}; K\big)\big] &= -1 + \frac{n}{n^2} \int_{\mathcal{X}} K(\boldsymbol{x}, \boldsymbol{x})\, \varrho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \frac{n^2 - n}{n^2} \times 1 \\ &= \frac{1}{n}\left[-1 + \int_{\mathcal{X}} K(\boldsymbol{x}, \boldsymbol{x})\, \varrho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right]\end{aligned}$$

There seemed to be confusion about the definition of discrepancy and what happens when a point is random.

Test 2 Scores (Regular and Make-up)

Number of Students: 35, Minimum: 0, Maximum: 100, Mean: 47.1, Median: 40
Standard Deviation: 29.7, Quartiles (Q1, Q3): (26, 73)

```
10|00000
 9|
 8|57
 7|35
 6|2
 5|
 4|000122359
 3|0338899
 2|0666
 1|38
 0|025
```

Best of 2 Test Scores

Number of Students: 36, Minimum: 14, Maximum: 100, Mean: 62.3, Median: 52
Standard Deviation: 26.3, Quartiles (Q1, Q3): (40, 86.5)

```
10|00000
 9|23
 8|45679
 7|34569
 6|
 5|022
 4|000223359
 3|03389
 2|6
 1|4
```

Weighted Average So Far (2 Tests and Assignments)

Number of Students: 36, Minimum: 49, Maximum: 100, Mean: 74.8, Median: 71
Standard Deviation: 14.5, Quartiles (Q1, Q3): (62.5, 87.5)

10	0
9	0222569
8	23344578
7	1112
6	1123456667
5	67789
4	9